

DEFLECTIONS OF THE VERTICAL AND GEOID HEIGHTS FROM GRAVITY GRADIENTS

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In the 20th century more than 60000 torsion balance measurements were made in Hungary. At present efforts are made to rescue the historical torsion balance data; today 24544 torsion balance measurements are available for further processing in computer database. Previously only the horizontal gradients of gravity were used by geophysicists, but there is a good possibility in geodesy to interpolate deflections of the vertical, and to compute geoid heights from curvature gradients of gravity.

First the theory of the interpolation method is discussed, than results of test computations are presented. We have selected a test area where all kind of torsion balance measurements are available at 249 points. There were 3 astrogeodetic points providing initial data for the interpolation, and 10 checkpoints for controlling the results. The size of our test area is about 750 km² and the average site distance of torsion balance data is 1.5 - 2 km. The standard deviations of geoid height and deflection of the vertical differences at checkpoints were about $\pm 1-3$ cm, and $\pm 0.6''$ respectively; which confirm that torsion balance measurements give good possibility to compute very precise deflections of the vertical and local geoid heights at least for flat areas.

Keywords: Geoid determination, deflection of the vertical, Torsion balance measurements, curvature gradients of gravity, inversion.

1. Fundamentals of the interpolation method

Let us consider distribution of deflections of the vertical in a small area where torsion balance measurements are available. Let computations be referred to a Cartesian system, having an arbitrary point P_0 within the examined area as origin. Let $+x$ and $+y$ be the axes of the system point to the north and to the east, respectively, and let axis z coincide with vertical direction at P_0 so that its positive branch points downwards.

Thereby, direction z at any point P_i of the concerned area is parallel to the z -axis through point P_0 , and the direction x_i to the tangent of astronomical meridian through point P_0 , as illustrated by the arbitrary point P_i in Fig. 1. The z -axis at point P_i being parallel to the vertical at origin P_0 , presumably, direction of vector g_i at point P_i does not coincide with direction z . In Fig.1. vector $\overline{P_iV}$ is, in fact, projection of vector g_i on plane xz , while vector $\overline{P_iH}$ is projection of component g_{xi} of vector g_i on the same plane. (There are negligible differences between vectors $\overline{P_iV}$ and g_i as well as $\overline{P_iH}$ and g_{xi}).

Be Φ_0 the astronomical latitude of point P_0 , and let $\Delta\Phi_i$ symbolize the angle between direction $\overline{P_iV}$ and z at point P_i , so the astronomical latitude of point P_i is:

$$\Phi_i = \Phi_0 + \Delta\Phi_i.$$

While, according to Fig.1: $-g_{xi} = g_i \sin \Delta\Phi_i$; it is to be written, for a small angle $\Delta\Phi_i$, as:

$$\Delta\Phi_i = -\frac{g_{xi}}{g_i}. \quad (1)$$

The same can be written for the variation of astronomic longitude in plane yz to:

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$$\Delta A_i \cos \Phi_i = \frac{g_{yi}}{g_i}. \quad (2)$$

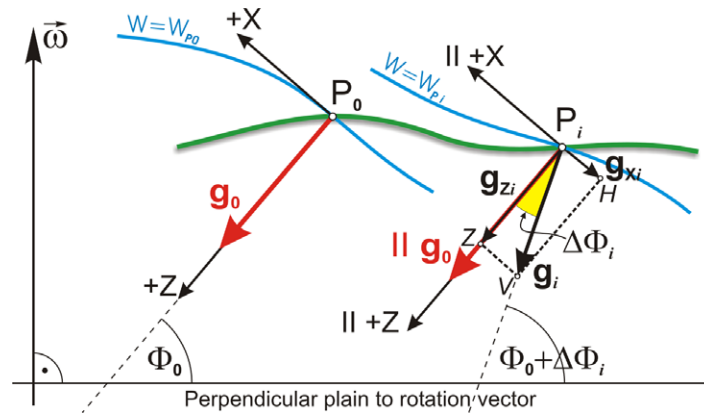


Fig. 1. Coordinate system for the interpolation

Equations (1) and (2) yield components N and E of the angle between geoid normal at points P_0 and P_i . Values $\Delta\Phi_k$ and ΔA_k for P_0 and some P_k may be determined in a similar way. These may be applied for writing differences between P_i and P_k :

$$(\Delta\Phi_k - \Delta\Phi_i) = -\left(\frac{g_{xk}}{g_k} - \frac{g_{xi}}{g_i}\right) \quad (3)$$

and

$$(\Delta A_k - \Delta A_i) \cos \tilde{\Phi} = -\left(\frac{g_{yk}}{g_k} - \frac{g_{yi}}{g_i}\right) \quad (4)$$

where $\tilde{\Phi}$ is the mean value of astronomical latitude between points P_i and P_k . By analogy with (1) and (2), (3) and (4) yield components N and E of the angle included by level surface normal at P_i and P_k .

By introducing notations $g_x = \frac{\partial W}{\partial x} = W_x$ and $g_y = \frac{\partial W}{\partial y} = W_y$ (W is the potential of Earth's real gravity field), Eqs (3) and (4) may be written as:

$$(\Delta\Phi_k - \Delta\Phi_i) = -\left(\frac{W_{xk}}{g_k} - \frac{W_{xi}}{g_i}\right) \quad (5)$$

and

$$(\Delta A_k - \Delta A_i) \cos \tilde{\Phi} = -\left(\frac{W_{yk}}{g_k} - \frac{W_{yi}}{g_i}\right) \quad (6)$$

respectively.

Level surfaces of the potential of normal gravity field, normal gravity, and directions of normal gravity vectors, in this relation, geodetic latitude and longitude of any point, termed normal geodetic latitude ${}_n\varphi$ and normal geodetic longitude ${}_n\lambda$, may be interpreted on the analogy of the Earth's real gravity field.

Relationships similar to (5) and (6) may be written between the variation of the gravity field direction in normal gravity field, that is, of normal geodetic coordinates ${}_n\varphi$ and ${}_n\lambda$ of points P_i and P_k , and the derivatives conform to potential of the normal gravity field (normal potential):

$$\Delta_n \varphi_k - \Delta_n \varphi_i = - \left(\frac{U_{xk}}{\gamma_k} - \frac{U_{xi}}{\gamma_i} \right) \quad (7)$$

and

$$(\Delta_n \lambda_k - \Delta_n \lambda_i) \cos \tilde{\varphi} = - \left(\frac{U_{yk}}{\gamma_k} - \frac{U_{yi}}{\gamma_i} \right) \quad (8)$$

where U is the normal potential, and γ_i , γ_k are the values of normal gravity at points P_i and P_k . Let us subtract Eqs (5) and (7), as well as (6) and (8) from each other:

$$(\Delta \Phi_k - \Delta_n \varphi_k) - (\Delta \Phi_i - \Delta_n \varphi_i) = - \left(\frac{W_{xk}}{g_k} - \frac{W_{xi}}{g_i} \right) + \left(\frac{U_{xk}}{\gamma_k} - \frac{U_{xi}}{\gamma_i} \right) \quad (9)$$

$$[(\Delta \lambda_k - \Delta_n \lambda_k) - (\Delta \lambda_i - \Delta_n \lambda_i)] \cos \tilde{\varphi} = - \left(\frac{W_{yk}}{g_k} - \frac{W_{yi}}{g_i} \right) + \left(\frac{U_{yk}}{\gamma_k} - \frac{U_{yi}}{\gamma_i} \right). \quad (10)$$

By definition, differences (9) and (10) between astronomic and normal geodetic latitudes and longitudes yield differences of components ξ and η of deflection of the vertical between points P_i and P_k :

$$\xi_k - \xi_i = - \left(\frac{W_{xk}}{g_k} - \frac{U_{xk}}{\gamma_k} \right) + \left(\frac{W_{xi}}{g_i} - \frac{U_{xi}}{\gamma_i} \right) \quad (11)$$

$$\eta_k - \eta_i = - \left(\frac{W_{yk}}{g_k} - \frac{U_{yk}}{\gamma_k} \right) + \left(\frac{W_{yi}}{g_i} - \frac{U_{yi}}{\gamma_i} \right). \quad (12)$$

Inside a limited area of size $0.5^\circ \times 0.5^\circ$, approximations $\tilde{\gamma} = \tilde{g}$ and $\tilde{\Phi} = \tilde{\varphi}$ are permissible – and so are single values \tilde{g} and $\tilde{\varphi}$ valid for all the area rather than between two neighboring points alone (Selényi 1953, Badekas and Mueller 1967) to be indicated simply by g and φ . Introducing notations

$$\Delta W = W - U \quad (13)$$

leads to equations:

$$g(\xi_k - \xi_i) = -\Delta W_{xk} + \Delta W_{xi}, \quad (14)$$

$$g(\eta_k - \eta_i) = -\Delta W_{yk} + \Delta W_{yi}. \quad (15)$$

Remind that in classic geodesy, deflection of the vertical is frequently interpreted as:

$$\xi = \Phi - \varphi$$

$$\eta = (A - \lambda) \cos \varphi$$

where Φ and A are astronomic coordinates, while φ and λ are geodetic (ellipsoidal) coordinates of point.

By physically interpreting the ellipsoid, serving as reference surface, as one level surface of the normal gravity field, than ellipsoidal and normal geodetic coordinates are related as $\varphi = \varphi - \kappa$ and $\lambda = \lambda - \lambda$ (κ is the difference of directions of the normal gravity field between point P on the earth surface and the ellipsoid surface along the normal plumb line at point P). The mentioned $0.5^\circ \times 0.5^\circ$ area, variation of κ is practically negligible (Magnitzki and Brovar 1964). Hence, (11) and (12) are also valid for the classical geodetic interpretation of deflection of the vertical. Thus, in the

following, when interpreting of deflection of the vertical it is needless to distinguish between the two conceptions, permitting to use the concept of deflection of the vertical in both interpretations.

Components of deflections of the vertical - more closely, their values multiplied by g , that is, horizontal components - seemed to be determined by first derivatives of the potential. While torsion balance measurements yield second derivatives

$$W_{\Delta} = \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \quad \text{and} \quad W_{xy} = \frac{\partial^2 W}{\partial x \partial y} .$$

Thus, the computation problem is essentially an integration to be solved by approximation.

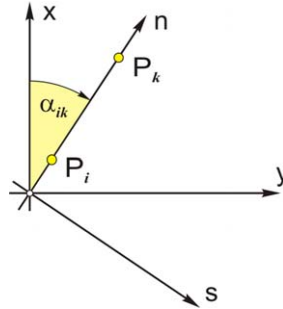


Fig. 2. $x, y \rightarrow n, s$ coordinate transformation

To this aim, first the coordinate transformation in Fig.2 will be performed, according to

$$\begin{bmatrix} n \\ s \end{bmatrix} = \begin{bmatrix} \cos \alpha_{ik} & \sin \alpha_{ik} \\ -\sin \alpha_{ik} & \cos \alpha_{ik} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Accordingly:

$$W_n = \frac{\partial W}{\partial n} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial n} = W_x \cos \alpha_{ik} + W_y \sin \alpha_{ik} \quad (16)$$

$$W_s = \frac{\partial W}{\partial s} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial s} = -W_x \sin \alpha_{ik} + W_y \cos \alpha_{ik}$$

while the second derivative:

$$\frac{\partial^2 W}{\partial n \partial s} = \frac{\partial^2 W}{\partial x \partial y} \cos 2\alpha_{ik} + \frac{1}{2} \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right) \sin 2\alpha_{ik} \quad (17)$$

This latter $W_{ns} = \frac{\partial^2 W}{\partial n \partial s}$ seems to result from torsion balance measurements, with the knowledge of azimuth α_{ik} of the direction connecting the two points examined.

Now, by integrating the left-hand side of (17) between limits n_i and n_k :

$$\int_{n_i}^{n_k} \frac{\partial^2 W}{\partial n \partial s} dn = \left(\frac{\partial W}{\partial s} \right)_k - \left(\frac{\partial W}{\partial s} \right)_i = W_{s_k} - W_{s_i} \quad (18)$$

If points P_i and P_k are close enough to let variation of second derivative W_{ns} be considered as linear, then integral (18) may be computed by trapezoid integral approximation formula:

$$\int_{n_i}^{n_k} \frac{\partial^2 W}{\partial n \partial s} dn = \frac{1}{2} \left[\left(\frac{\partial^2 W}{\partial n \partial s} \right)_i + \left(\frac{\partial^2 W}{\partial n \partial s} \right)_k \right] (n_k - n_i) = \frac{1}{2} [(W_{ns})_i + (W_{ns})_k] n_{ik} \quad (19)$$

where $n_{ik} = n_k - n_i$ is the distance between points P_i and P_k . On the other hand, by applying transformation (16), integral (18) yields:

$$W_{sk} - W_{si} = -(W_{xk} - W_{xi}) \sin \alpha_{ik} + (W_{yk} - W_{yi}) \cos \alpha_{ik} . \quad (20)$$

Similar expression can be written for the potential U of normal gravity field:

$$U_{sk} - U_{si} = -(U_{xk} - U_{xi}) \sin \alpha_{ik} + (U_{yk} - U_{yi}) \cos \alpha_{ik} . \quad (21)$$

Subtracting (21) from (20) yields variation $G_{ik} = g \Delta \mathcal{G}_{ik}$ of horizontal force component between points P_i and P_k in direction n . By taking (13) into consideration:

$$G_{ik} = (-\Delta W_{xk} + \Delta W_{xi}) \sin \alpha_{ik} - (-\Delta W_{yk} + \Delta W_{yi}) \cos \alpha_{ik}$$

after substituting (14) and (15):

$$G_{ik} = g(\mathcal{G}_k - \mathcal{G}_i) \mathcal{G}_{ik} = g(\xi_k - \xi_i) \sin \alpha_{ik} - g(\eta_k - \eta_i) \cos \alpha_{ik} \quad (22)$$

The left-hand side of (22) may be computed by using (19). When using notation (13):

$$G_{ik} = \frac{1}{2} [(\Delta W_{ns})_i + (\Delta W_{ns})_k] n_{ik} \quad (23)$$

with ΔW_{ns} to be computed from (17):

$$\Delta W_{ns} = \Delta W_{\Delta} \sin 2\alpha_{ik} + \Delta W_{xy} \cos 2\alpha_{ik} \quad (24)$$

where $\Delta W_{\Delta} = W_{\Delta} - U_{\Delta}$ and $\Delta W_{xy} = W_{xy} - U_{xy}$. Remind that W_{Δ} and W_{xy} are curvature gradients obtainable from torsion balance measurements, while U_{Δ} and U_{xy} are curvature gradients of the normal gravity field. $U_{\Delta} = 10.26 \cos^2 \varphi$ and $U_{xy} = 0$ referred to the GRS-80 in Eötvös Unit (1 Eötvös Unit = 1E = $10^{-9} s^{-2}$), (Torge 1989). Now, by substituting (24) into (23):

$$G_{ik} = \frac{n_{ik}}{4} \left[(\Delta W_{\Delta_i} + \Delta W_{\Delta_k}) \sin 2\alpha_{ik} + (\Delta W_{xy_i} + \Delta W_{xy_k}) \cos 2\alpha_{ik} \right] \quad (25)$$

which, compared to (22) yields the basic equation, relating the variation of components of deflections of the vertical between two points to curvature gradients from torsion balance measurements:

$$\begin{aligned} (\xi_k - \xi_i) \sin \alpha_{ik} - (\eta_k - \eta_i) \cos \alpha_{ik} = \\ \frac{n_{ik}}{4g} \left[(\Delta W_{\Delta_i} + \Delta W_{\Delta_k}) \sin 2\alpha_{ik} + (\Delta W_{xy_i} + \Delta W_{xy_k}) \cos 2\alpha_{ik} \right] . \end{aligned} \quad (26)$$

Based on (26) there are $2n$ unknowns in case of an arbitrary interpolation net of n points. E.g. in case of a simple triangulation chain the n points form $n-2$ triangles with $2n-3$ triangle sides; so there are $2n-3$ equations for the $2n$ unknowns. For unambiguous solution further information are required (Völgyesi 1993, 1995). Two or more fixed points are needed where the deflections of the vertical are given.

2 Computation of geoid heights from deflections of the vertical

The basic principle of astronomical levelling gives us a definite mathematical relationship between geoid undulations and deflections of the vertical (Völgyesi 2001b). According to the notations of Fig. 3 we get

$$dN = \mathcal{G} ds$$

where \mathcal{G} is the *Pizzetti-type* deflection of the vertical in the azimuth α . Between any points P_i and P_k the geoid height change is

$$\Delta N_{ik} = \int_i^k \mathcal{G}(s) ds \quad (27)$$

where $\mathcal{G}(s)$ is the function of deflection of the vertical in the azimuth α_{ik} , and s is the distance between the two points (Biró 1985).

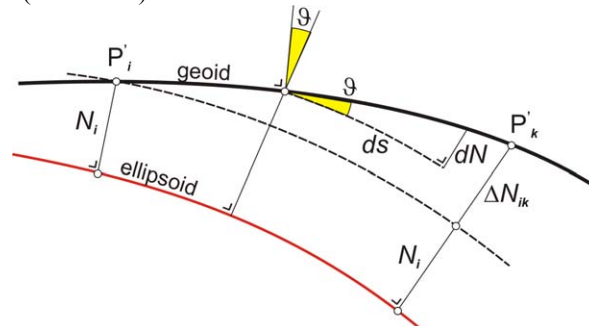


Fig. 3. Basic principle of astronomical levelling

If P_i and P_k are close together and $\mathcal{G}(s)$ is a linear function between these points the integral (27) can be evaluated by a numerical integration (Völgyesi 1998, Tóth Gy, Völgyesi L 2002):

$$N_k - N_i = \left(\frac{\xi_i + \xi_k}{2} \cos \alpha_{ik} + \frac{\eta_i + \eta_k}{2} \sin \alpha_{ik} \right) s_{ik} \quad (28)$$

because of $\mathcal{G} = \xi \cos \alpha + \eta \sin \alpha$. Using the original torsion balance points directly for the geoid computation instead of regular grid points is advantageous (Völgyesi 2001b). For an unambiguous solution one or more fixed points are needed where the initial geoid heights are given.

3 Torsion balance measurements

In the 20th century more than 60000 torsion balance measurements were made in Hungary (Szabó, 1999). Generally only the horizontal gradients of gravity W_{zx} and W_{zy} were developed for the purposes of geophysical prospecting, and the curvature values W_{Δ} and W_{xy} , which are very important for geodetic purposes, were left unprocessed. Unfortunately a small part of field books of torsion balance measurements were lost up to now, but the main part of measurements is still available (Völgyesi, 2001a). At present time serious efforts are going on for rescuing the historical torsion balance measurements, researchers of ELGI are creating the data of old field books to computer database. At present 24544 torsion balance measurements are available for further processing in computer database. Location of these torsion balance points can be seen on Fig. 4.

The name of the torsion balance station, measurement year, geographical latitude φ , geographical longitude λ , horizontal and curvature values W_{zx} , W_{zy} , W_{Δ} , $2W_{xy}$ and terrain reductions of these gradients are stored in the database for each station.

Test computations were performed in an area extending over some 750 km^2 (a rectangle shows the position of this test area in Fig. 4) and well measured by torsion balance, where both topographic conditions and the density of torsion balance measurements and astrogeodetic stations reflects average conditions in Hungary; and there was a possibility to check calculations because astrogeodetic and astrogravimetric data were available. In Figs. 5 and 6 curvature gradients W_{Δ} and $2W_{xy}$ measured by torsion balance are visualized in the test area.

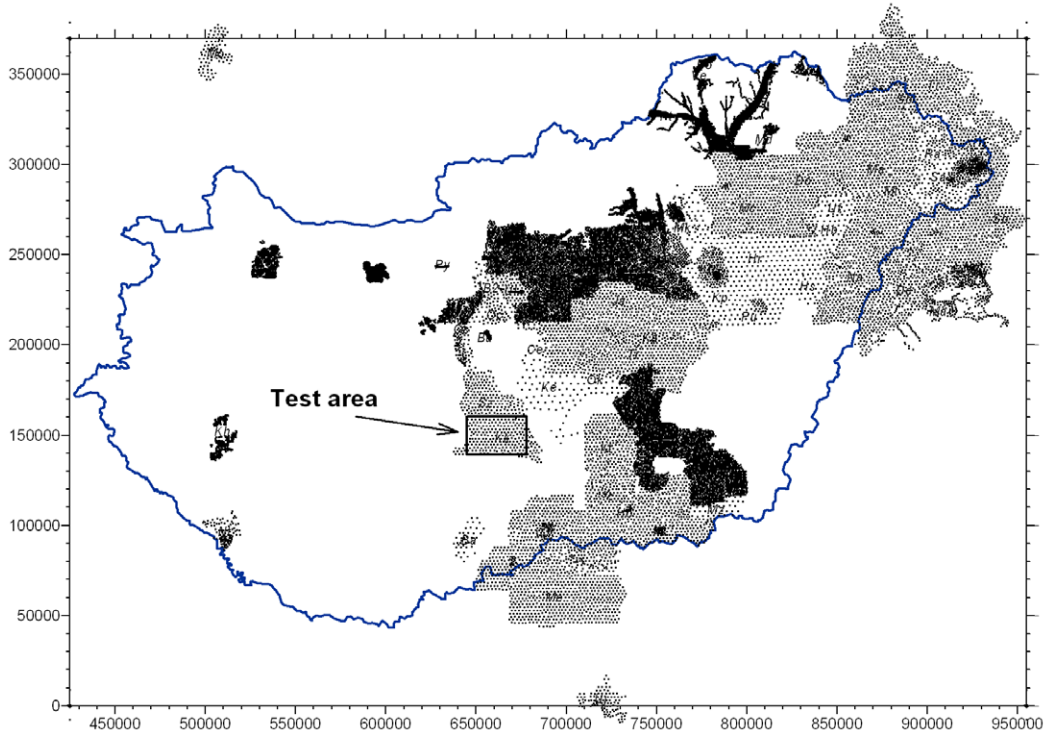


Fig. 4. Location of 24544 torsion balance points having been developed to computer database till now

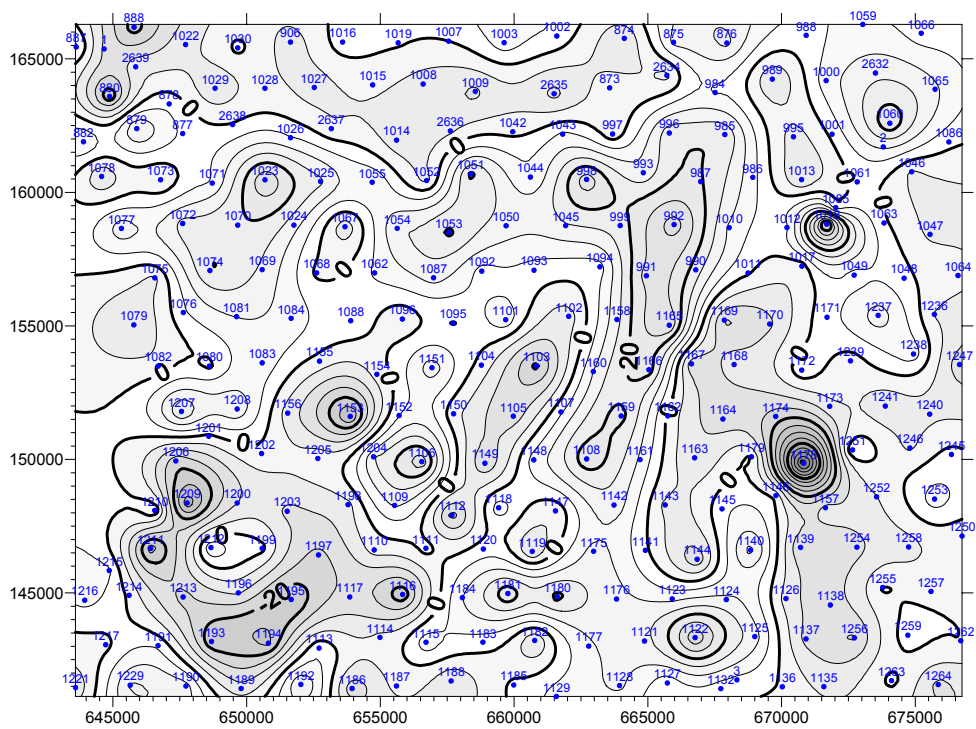


Fig. 5. Isoline map of curvature data W_A on the test area

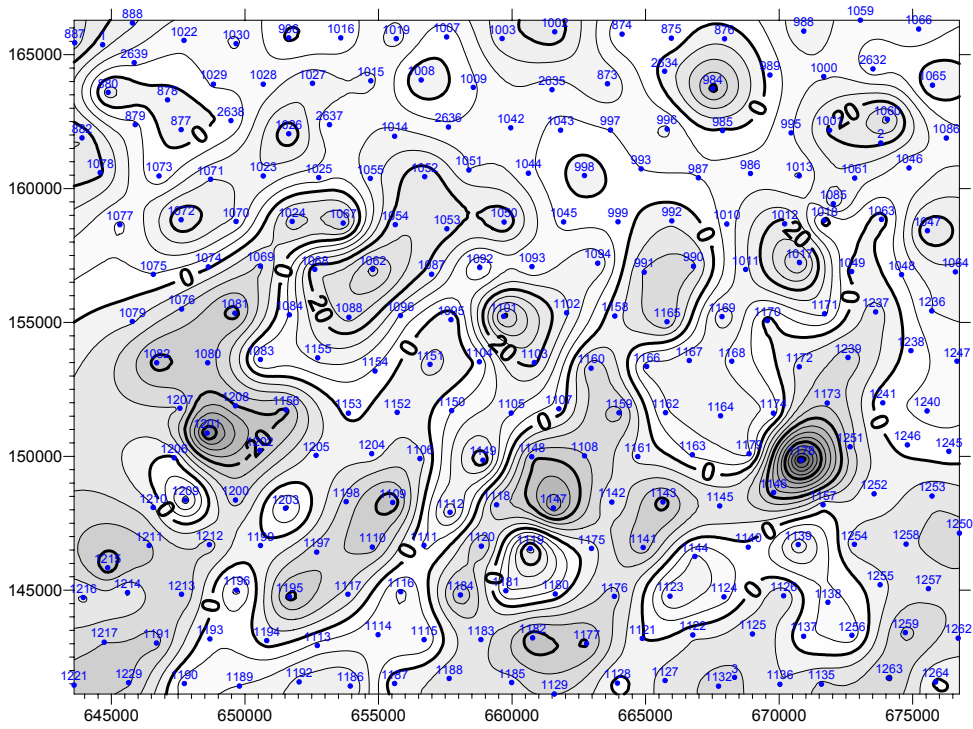


Fig. 6. Isoline map of curvature data $2W_{xy}$ on the test area

4 Test computations

Software was developed for computations which can be used to determine deflections of the vertical and geoid undulations. Input-output window of this FuggoOrt software can be seen in Fig. 7.

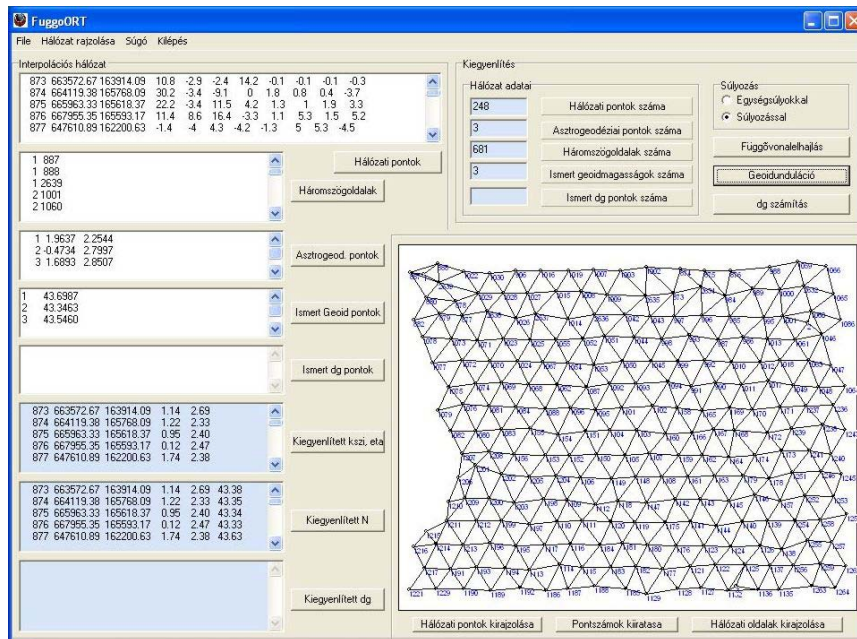


Fig. 7. Input-output window of the interpolation software

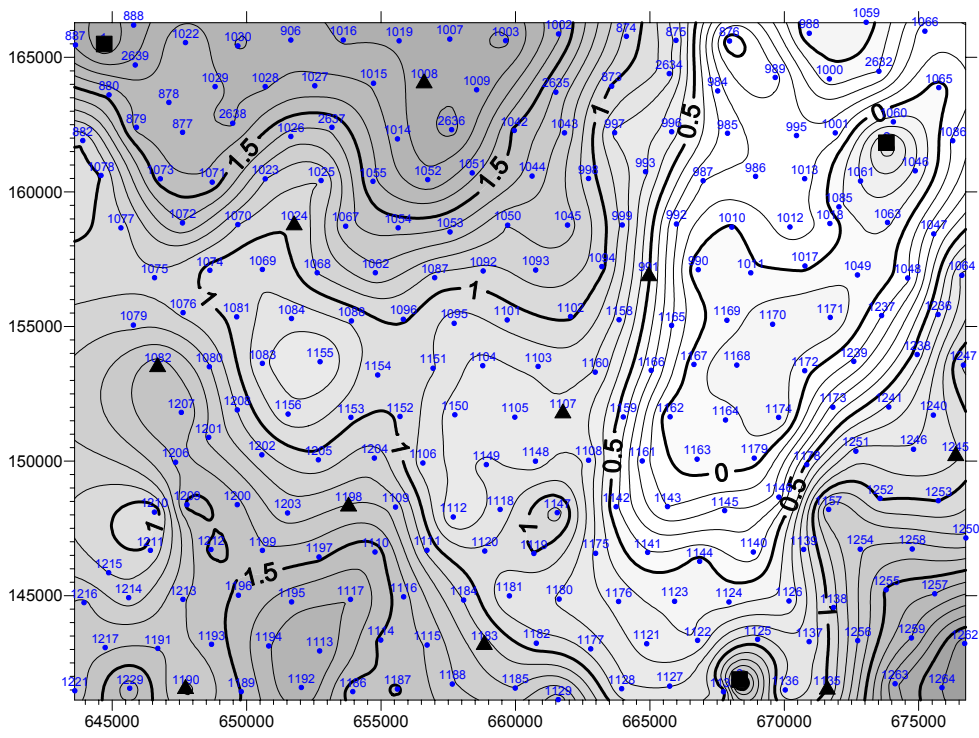


Fig. 8. Isoline map of ξ component of interpolated deflections of the vertical

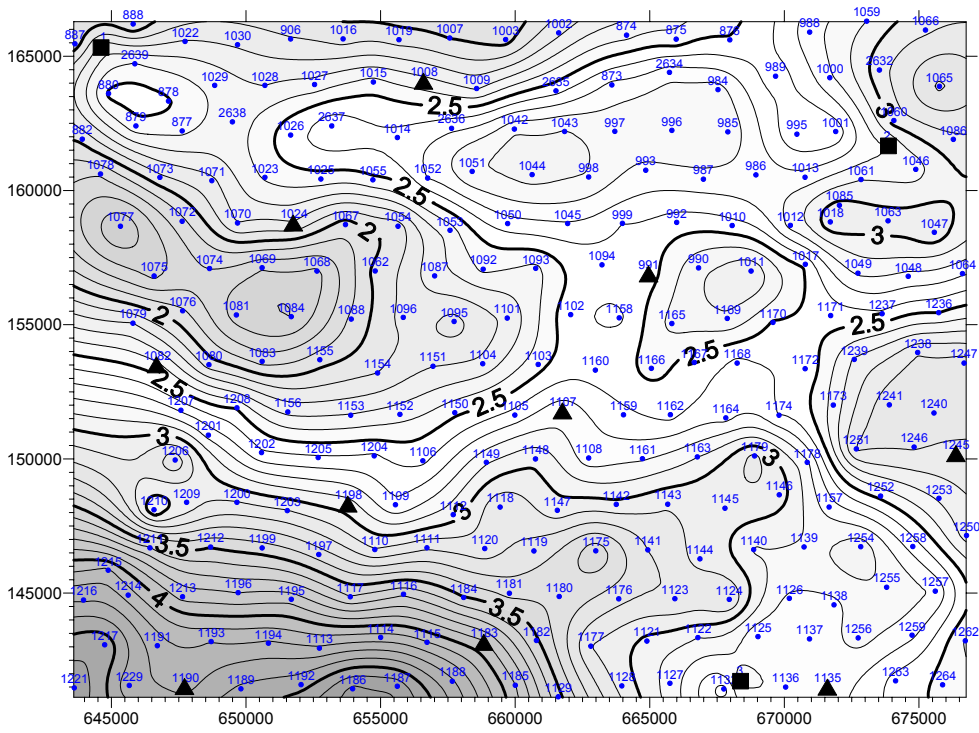


Fig. 9. Isoline map of η component of interpolated deflections of the vertical

Test computations were performed in the test area extending over about 750 km^2 . There are 248 torsion balance stations, and 13 points (3 astrogeodetic, and 10 astrogravimetric points) among them where ξ , η and N values are known referring to the *GRS80* system. The 3 astrogeodetic points indicated with squares in Figs. 8, 9 and 10 were used as initial (fixed) points of interpolations and the 10 astrogravimetric points indicated with triangles in Figs. 8, 9 and 10 were used for checking of computations.

The interpolation network has 248 points in all and 245 of these are points with unknown deflections. Since there are two unknown components of deflection of the vertical at each point there are 498 unknowns for which 681 equations can be written.

Isoline maps of interpolated ξ and η components of deflections of the vertical can be seen in Figs. 8 and 9. Standard deviations computed at checkpoints are: $m_\xi = \pm 0.60''$ and $m_\eta = \pm 0.65''$.

Based on the computed deflection of the vertical components, using Eq (28) geoid computations were carried out. The determined geoid map can be seen in Fig. 10. Geoid height differences computed at checkpoints are about $\pm 1\text{-}3 \text{ cm}$.

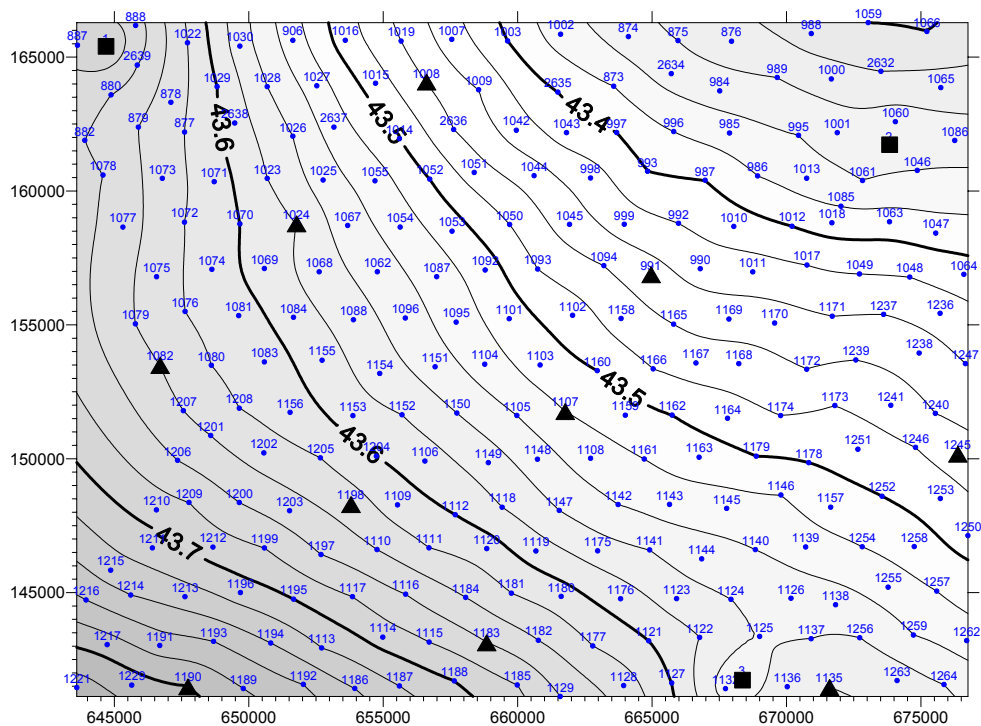


Fig. 10. Isoline map of geoid undulations

5 Inversion reconstruction of gravity potential

Further investigation is started for the inversion reconstruction of gravity potential based on torsion balance measurements. This method gives an additional possibility to determine deflections of the vertical based on torsion balance measurements. Based on the preliminary investigations accuracy of this method may be better than the accuracy of the former interpolation methods, and it is possible to avoid some types of earlier interpolation problems, like e.g. accuracy dependence of the geometrical configuration of the interpolation net (Völgyesi, 1995).

Based on the experimental computations by synthetic data, inversion algorithm seems to be remarkably stable (Dobróka, Völgyesi 2005). This method gives possibility to determine the potential

function by a common inversion using both torsion balance measurements and deflections of the vertical data at a few given points. Many important features of the gravity field (e.g. deflections of the vertical at the unknown points) can be determined from this reconstructed potential function.

Summary

Our test computations confirm the fact that for large and more or less flat territories deflections of the vertical and geoid heights of acceptable accuracy can be computed from torsion balance measurements. Reliability of interpolated deflection of the vertical and geoid height values are about $\pm 0.6''$ and $\pm 1-3$ cm respectively. In the next future waiting for the very precise measurements of the promising new modern gradiometers our method and software are ready for use to evaluate these further measurements.

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