The Eötvös spherical horizontal gradiometric boundary value problem - gravity anomalies from gravity gradients of the torsion balance

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Abstract. The closed-form solution of a certain overdetermined fixed horizontal gradiometric boundary value problem is given in spherical, constant radius approximation. The gravity anomalies are expressed in terms of surface integrals of inner products of the boundary data with vector and tensor valued Green’s functions. The boundary data are the four components \( \Gamma_{xy}, \Gamma_{xx} - \Gamma_{yy}, \Gamma_{xz}, \Gamma_{yz} \) of the gravity tensor, measured by the Eötvös torsion balance (hence the name for the problem), grouped into a vectorial and tensorial combination. The corresponding Green’s functions can multiplicatively be decomposed into an azimuth dependent and isotropic part (kernel). The spatial truncation characteristics of the surface integrals are examined and a practical method of evaluating their spectral truncation coefficients is provided. Finally a method is discussed to validate the software that has been developed by using synthetic gravity data.

Keywords: gravity gradient tensor — torsion balance — geodetic boundary value problem — vector and tensor spherical harmonics — truncation coefficients

1 Introduction

The formulation and solution of geodetic boundary value problems has a long history in geodesy. The target of those boundary value problems had been to determine the gravity field exterior to the Earth’s topographic surface (or some other reference surface) in terms of functionals of the gravitational potential on the Earth’s physical surface. These functionals can be zero-order derivative: Dirichlet problem; first-order derivative: horizontal and vertical boundary value problems; and second-order derivative: gradiometric boundary value problems. Our target here are gradiometric boundary value problems.

The gravity gradient tensor has nine elements in the \( \mathbb{R}^{3 \times 3} \) linear (tensor) gravity space but only five of them are independent, since it must be a symmetric and trace-free second rank tensor in the outer space. When dealing with gradiometric boundary value problems we have to specify exactly what components of the gravity gradient tensor are considered as measured on or above the Earth’s topographic surface (or on some suitable reference surface).

Another question is the choice of the reference frame of gradient tensor observables. A common choice is the local moving unimodular (orthogonal) triad, pointing (astronomical) North, East, Down. It is customary to define the various tensor components with respect to this frame as vertical-vertical, mixed horizontal-vertical and horizontal-horizontal components, respectively.

A third question is which particular combinations of the individual tensor components are considered as observables in the gradiometric boundary value problem. The vertical and diagonal terms are the most commonly used ones (see for example Heck, 1979; Petrovskaya and Zieliński, 1998; Brovelli and Sansò, 1990). Our contribution is about the classical horizontal-vertical and horizontal-horizontal combinations observed by the Eötvös torsion balance (Selényi, 1953).

Our motivation for formulating and solving gradiometric boundary value problems of torsion balance gradient observables is twofold. First, we are currently reassessing torsion balance data for high precision geoid determination in Hungary (Tóth et al., 2002). Second, this is by itself a very interesting kind of boundary value problem of overdetermined type which has an exact, closed-form Green function solution, in this respect similar to the Stokes problem. Also, our contribution can be considered as one step further in the direction of the set-up and solution of boundary value problems on the Interna-
Our main target is, therefore, the formulation and solution of the following gradiometric boundary value problem: given horizontal-vertical and horizontal-horizontal gravity gradient combinations on the sphere $S^2_R$, find the harmonic gravitational disturbing potential (incremental potential), and here more specifically: gravity anomalies with respect to a suitable reference potential (gravity) in the 3-D space external to $S^2_R$.

Our second target comes from a practical necessity: Since our observable tensor quantities are almost always restricted to a limited domain in space, there is a need to quantitize exactly the consequences of such truncation on the solution. This will be achieved via a set of truncation formulas, transformed into the spectral domain. A similar problem is the well-known truncation problem of the Stokes integral. Our aim here is to introduce a practical evaluation method of spectral truncation coefficients of the isotropic Green functions and a close examination will reveal very interesting truncation properties of these functions.

Finally, it is a good practical check of the correctness of our solution as well as the software under development to use synthetic gravity gradient data for the evaluation of gravity anomalies. We discuss briefly this possibility in the last part of our paper.

The remainder of the paper is organized as follows: In Section 2 we set-up the two incremental gravity gradient combinations, a vectorial one for spherical vertical-horizontal gradients and a tensorial one for horizontal-horizontal combinations measured by the torsion balance. Section 3 outlines the representation of the above combinations in spheroidal vector and shear tensor spherical harmonics. This leads to the least squares solution of the overdetermined problem in the spectral domain. In Section 4 the closed-form solution of the overdetermined problem for gravity anomalies is discussed in terms of two Green functions — a vector and a tensor valued one. This result is achieved partially via powerful addition theorems of vector and tensor harmonics, leading to multiplicative decompositions of Green functions into a scalar (kernel) and vector (resp. tensor) valued part. Section 5 discusses the spatial truncation characteristics of these scalar valued, isotropic kernels. A particular highlight is the surprisingly different truncation behaviour of the kernels for incremental potential versus gravity anomaly. Finally, in Section 6 the software validation process by synthetic gravity gradient field data is outlined and a summary is given on the main results of this contribution.

This paper is based on the works of many researchers, just to mention a few of them: Heck (1991); Sansò (1993); Rummel (1997); Grafarend (2001) and van Gelderen and Rummel (2001). Finally we would like to mention L. Eötvös, the outstanding Hungarian physicist, whose ingenious device, the torsion balance made it first possible to measure gravity gradients with unprecedented $10^{-9} \text{s}^{-2}$ accuracy. Therefore, the special kind of overdetermined gradiometric boundary value problem, which is discussed in this paper, is properly called after his name the Eötvös gradiometric boundary value problem for gravity anomaly.

## 2 The set-up of incremental gradient and shear combinations of the gravity gradient tensor — torsion balance observables

To set up the Eötvös gradiometric boundary value problem we face with several important issues. First, we have to specify a suitable reference potential field and decompose the gravity gradient tensor field $\Gamma(x), \Gamma \in \mathbb{R}^{3 \times 3}, x \in \mathbb{R}^3$ into a reference part and a disturbing part. Second, the torsion balance incremental tensor observables have to be grouped into suitable combinations (called incremental Eötvös observables). The idea here is that these combinations will transform in a specific way and as we will see later in Section 3, they will allow for a straightforward representation in the spectral domain of vector and tensor spherical harmonics.

A third issue would be the downward continuation problem of incremental Eötvös observables from the topographic surface to the reference sphere $S^2_R$. This problem is not discussed here — we make the assumption that the incremental Eötvös observables are given on this reference sphere.

The coordinate frame we will be working in is the local spherical (moving) orthonormal frame $\{e_1, e_2, e_3 \mid P\} = \{e_\alpha, e_1, -e_\tau \mid P\}$ at $P$, illustrated in Fig. 1. This frame is oriented spherical North, East and Down, respectively, thus forming a right-handed basis. We must be careful to select the reference frame of the gravity gradient tensor $\Gamma(P)$. In terrestrial gradiometry the gradient tensor field is supposed to be defined at $P$ in the local astronomical triad, which is aligned with the direction of the geogravity vector $\mathbf{g}(P)$ and astronomical North, East. Therefore, a rotation $\mathbf{R} \Gamma \mathbf{R}^T$ is to be performed on $\Gamma$ to transform it to the local spherical triad $\{e_1, e_2, e_3 \mid P\}$ by a suit-
able rotation matrix $\mathbf{R}$ (and its transpose $\mathbf{R}^T$, according to the transformation law of tensors). However, it was explained by Hein (1981), that the tensor elements under such a transformation will change only by an amount less than 0.1 E. This change can be neglected in terrestrial gradiometry, since the measurement accuracy of torsion balance gravity gradients is at the level of $\pm 1 - 2$ E.

The reference potential field $w(P) = w(\phi, \lambda, r)$ is chosen suitably (for example a Somigliana-Pizzetti type field) to introduce the incremental potential field

$$\delta w(P) := W(P) - w(P)$$

where the scalar-valued geopotential field is denoted by $W(P)$ at point $P$.

In the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid P\}$ local spherical orthonormal frame we will define the *incremental gravity gradient tensor field* (the incremental Eötvös tensor) $\delta \Gamma(P)$ in the following way

$$\delta \Gamma(P) := \Gamma(P) - \gamma(P).$$

(2)

We note that Eq. (2) implies that the position of $P$ is known with sufficient accuracy to set up a fixed gradiometric boundary value problem. For terrestrial gradiometry this assumption is justified. The incremental Eötvös tensor has nine elements $\delta \Gamma_{ij}(P) := \Gamma_{ij}(P) - \gamma_{ij}(P), \quad i, j \in \{1, 2, 3\}$. The reference tensor part, $\gamma(P)$ is defined in spherical coordinates $P \sim (\phi, \lambda, r)$ as

$$\gamma(\phi, \lambda, r) := \nabla \otimes \nabla w(\phi, \lambda, r),$$

(3)

where $\nabla$ denotes the gradient operator in $\mathbb{R}^3$ and $\otimes$ stands for the outer (tensor) product of two vector spaces. The incremental gravity gradient tensor is similarly

$$\delta \Gamma(\phi, \lambda, r) = \nabla \otimes \nabla \delta w(\phi, \lambda, r)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \delta \Gamma_{ij}(\phi, \lambda, r).$$

(4)

With the notation $\partial_i := \partial/\partial x_i$ the incremental gravity gradient tensor $\delta \Gamma(P)$ of Eq. (4) has the following components in $\mathbb{R}^{3\times3}$

$$\delta \Gamma_{ij}(P) = \partial_i^2 \delta w(P), \quad i, j \in \{1, 2, 3\}.$$  

(5)

As we have completed our first step and decomposed the gravity gradient tensor $\Gamma$ into a reference and incremental part in Eqs. (2), (4) and (5) we will establish two combinations from torsion balance observables $\delta \Gamma_{ij} = \delta \Gamma_{11}, \delta \Gamma_{12}, \delta \Gamma_{13}, \delta \Gamma_{23}, \delta \Gamma_{33}$, a vector- and a tensor-valued one. We will call these combinations the *incremental Eötvös observables*.

The first combination is a vector in the local tangent space at $P$, $T_P \mathbb{S}_r^2$. This vector, $\delta \Gamma^{(1)} \sim (\delta \Gamma_{13}, \delta \Gamma_{23})$ is formed from the incremental Eötvös tensor components of Eq. (5) in the following way:

$$\delta \Gamma^{(1)}(P) := \delta \Gamma_{13}(P) \mathbf{e}_1 + \delta \Gamma_{23}(P) \mathbf{e}_2.$$ 

(6)

This vector is the horizontal gradient of the radial component of the gravity disturbance vector

$$\delta \gamma(P) := -\nabla_r \delta w(P),$$

(7)

and therefore it is proper to call it the *incremental horizontal gradient vector* or the *incremental gradient combination*. To prove this statement we write Eq. (5) explicitly for the components of the $\delta \Gamma^{(1)}$ combination

$$\delta \Gamma_{13}(P) = \partial_{r_{13}}^2 \delta w(P)$$

$$= -\partial_r \frac{1}{r} \partial_\phi \delta w(P) = \frac{1}{r} \partial_\phi \delta \gamma(P)$$

$$\delta \Gamma_{23}(P) = \partial_{r_{23}}^2 \delta w(P)$$

$$= -\partial_r \frac{1}{r \cos \phi} \partial_\lambda \delta w(P) = \partial_r \frac{1}{r \cos \phi} \partial_\lambda \delta \gamma(P).$$

(8)

If we now introduce the spherical surface gradient operator

$$\nabla_{surf} := \mathbf{e}_\phi \partial_\phi + \frac{\mathbf{e}_\lambda}{\cos \phi} \partial_\lambda$$

(9)
we see immediately from Eqs. (6), (8) and (9) that
\[
\delta \mathbf{T}^{(1)}(P) = \frac{1}{r} \mathbf{\nabla}_{\text{surf}} \delta \gamma(P)
\]  
(10)
holds and hence \(\delta \mathbf{T}^{(1)}\) is really the spherical horizontal gradient of the gravity disturbance defined by Eq. (7). The gradient combination \(\delta \mathbf{T}^{(1)}\) is connected to the incremental potential \(\delta \nu\) by the following covariant second-order horizontal differential operator \(\mathbf{\nabla}^{(1)}\) as
\[
\delta \mathbf{T}^{(1)}(P) = \mathbf{\nabla}^{(1)}(\delta \nu(P)).
\]
(11)
This relation follows directly from the following definition
\[
\mathbf{\nabla}^{(1)}(P) := -\partial_r \frac{1}{r} \mathbf{\nabla}_{\text{surf}}(P).
\]
(12)
We note that the incremental horizontal gravity gradient \(\delta \mathbf{T}^{(1)}\) transforms under an \(e\)-rotation \(R_3(e)\) of the \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} | P\) frame around the \(\mathbf{e}_3\) axis in the local tangent space \(T_P S^2\) as a covariant vector quantity and it can be represented by a vector \(\delta \mathbf{T}^{(1)}\) in this local tangent space (Fig. 2).

![Fig. 2: The incremental gradient combination \(\delta \mathbf{T}^{(1)}\) can be represented by a vector in gravity space (local tangent space, \(T_P S^2\)) with its magnitude \(||\delta \mathbf{T}^{(1)}||:\) and direction \(\tan \alpha = \delta \Gamma_3/\delta \Gamma_2\).](image)

Now we set up the second combination of the incremental Eötvös observables. The two element set \((\delta \Gamma_\Delta, \delta \Gamma_{12})\) can be called the curvature variation (Heiland, 1929) or curvature gravity gradient. It is also called the horizontal directional tendency (HDT) after Eötvös, who used the German word "Richtkraft" (Dransfield, 1994). The \((\delta \Gamma_\Delta, \delta \Gamma_{12})\) combination does not transform like a vector quantity. Instead, we introduce the following tensor-valued incremental gravity gradient combination
\[
\delta \mathbf{T}^{(2)}(P) := \frac{1}{\sqrt{2}} \begin{pmatrix} \delta \Gamma_\Delta & -2\delta \Gamma_{12} \\ -2\delta \Gamma_{12} & -\delta \Gamma_\Delta \end{pmatrix}(P)
\]
(13)
as a 2-rank symmetric trace free (STF) tensor formed by the horizontal-horizontal observables of the torsion balance. Now again this incremental tensor-valued combination can be connected to the incremental potential \(\delta \nu\) by a suitable tensor-valued covariant second-order differential operator \(\mathbf{\nabla}^{(2)}\). In terms of second-order covariant differential operators \(\nabla^2_{\phi \phi} - \nabla^2_{\lambda \lambda}, \nabla^2_{\phi \lambda}\), we can define it as
\[
\mathbf{\nabla}^{(2)} := \frac{1}{\sqrt{2}} \begin{pmatrix} \nabla^2_{\phi \phi} - \nabla^2_{\lambda \lambda} & -2\nabla^2_{\phi \lambda} \\ -2\nabla^2_{\lambda \phi} & -(\nabla^2_{\phi \phi} - \nabla^2_{\lambda \lambda}) \end{pmatrix}.
\]
(14)
Here \(\nabla_\mu\) denotes covariant differentiation with respect to \(\mu\) on the 2-sphere \(S^2\). We note that the second-order covariant differential operators (components of \(\mathbf{\nabla}^{(2)}\)) have the following explicit forms on \(S^2\) in terms of spherical surface coordinates \(\phi, \lambda\) (see for example Rummel, 1997)
\[
\nabla^2_{\phi \phi} - \nabla^2_{\lambda \lambda} = \partial^2_{\phi \phi} + \tan \phi \partial_\phi \partial_\phi - \cos^{-2} \phi \partial^2_{\lambda \lambda}
\]
\[
-2\nabla^2_{\lambda \phi} = 2 \cos^{-1} \phi \partial_\lambda (\partial_\phi + \tan \phi).
\]
(15)
(16)
To summarize the above development, the incremental tensor-valued gravity gradient combination \(\delta \mathbf{T}^{(2)}\) can be expressed by acting with \(\mathbf{\nabla}^{(2)}\) on the incremental potential field \(\delta \nu(P)\) as
\[
\delta \mathbf{T}^{(2)}(P) = \frac{1}{r^2} \mathbf{\nabla}^{(2)} \delta \nu(P).
\]
(17)
The tensor-valued incremental gravity gradient combination \(\delta \mathbf{T}^{(2)}\) has a nice geometrical analogue, the shear tensor in the 2-D Euclidean plane \(\mathbb{R}^2\). The displacement field \((\Delta x, \Delta y)\) due to an infinitesimal shear with magnitude \(\gamma\) and direction \(\alpha\) can be characterized by the following expression (see also Fig. 3)
\[
\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \gamma \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{S}(\gamma, \alpha) \begin{pmatrix} x \\ y \end{pmatrix}
\]
(18)
with the STF shear tensor
\[
\mathbf{S}(\gamma, \alpha) := \begin{pmatrix} \gamma \cos 2\alpha & -\gamma \sin 2\alpha \\ -\gamma \sin 2\alpha & -\gamma \cos 2\alpha \end{pmatrix}.
\]
Fig. 3: An infinitesimal shear at azimuth $\alpha$ with magnitude $\gamma$ generates the displacement field $(\Delta x, \Delta y)$ on $\mathbb{R}^2$, which can be characterized by Eq. (18).

It is easy to see that any shear tensor $S(\gamma, \alpha)$ can be additively decomposed into the following linear combination

$$S(\gamma, \alpha) = \gamma S^0 \cos 2\alpha + \gamma S^\pi \sin 2\alpha,$$  

(19)

where $S^0 := S(1,0)$ and $S^\pi := S(1,\pi/4)$ are the two orthogonal basis tensors of shear.

The transformation law of the $\delta\Gamma^{(2)}$ tensorial combination under rotation by an angle $\alpha$ is exactly as (19), that is

$$\delta\Gamma^{(2)*} = R(\alpha) \delta\Gamma^{(2)} R'(\alpha) = \delta\Gamma^{(2)} \cos 2\alpha + \delta\Gamma^{(2)\pi} \sin 2\alpha,$$  

(20)

where $\delta\Gamma^{(2)\pi}$ is the positive $\pi/4$-rotated version of $\delta\Gamma^{(2)}$. This means that any tensor $\delta\Gamma^{(2)}$ can be decomposed into the linear combination of two orthonormal basis tensors, one tensor is arbitrary, the other basis tensor is rotated by the positive angle $\pi/4$ clockwise. In view of all the above we will call the $\delta\Gamma^{(2)}$ combination the incremental tensor-valued shear combination.

Now we have completed the second step in this Section since we have the gradient combination vector (Eq. 6) and shear combination tensor (Eq. 13) as two particular combinations of the incremental Eötvös observables. These combinations are suitable since we will see in the next Section how easy it is to transform these observable combinations to the spectral domain.

Finally we introduce the scalar gravity anomaly $\Delta y$ as the modulus of the gravity anomaly vector

$$\Delta \tilde{y}(P) := \tilde{\Gamma}(P) - \tilde{\gamma}(\hat{P}),$$

(21)

$\Delta y := \| \Delta \tilde{y} \|$, $\tilde{\gamma}(P)$ being the geogravity vector at $P$ and $\tilde{\gamma}(\hat{P})$ is the normal (reference) gravity vector at the approximate position $\hat{P}$ of $P$. We follow the notation $\Delta y$ for the scalar gravity anomaly of Heck (1991) to be consistent in our lettering, and we have used the tilde (‘) notation to distinguish vector-valued gravity field quantities from the tensor-valued ones $\Gamma$ and $\gamma$.

To conclude with our set-up we express the target of the Eötvös gradiometric boundary value problem, scalar gravity anomaly $\Delta y$ in terms of the Stokes operator $\nabla_{S^g}$ acting on the incremental potential $\delta w$

$$\Delta \tilde{y}(P) := \nabla_{S^g} \delta w(P) = \left( -\frac{2}{r} - \frac{\partial}{\partial r} \right) \delta w(P).$$

(22)

We can now summarize the set-up of the horizontal-vertical and horizontal-horizontal components of the incremental gravity gradient tensor. In the spherical geometry and gravity space, the observables of the torsion balance are represented in the spherical local reference frame $(e_0, e_1, -e_2 | P)$ at the point $P$ in terms of the two incremental combinations, the gradient combination vector $\delta\Gamma^{(1)}$, Eq. (6) and the shear combination tensor, Eq. (13).

3 Complex vector and tensor spherical harmonics — set-up and solution in the spectral domain

In order to set up the Eötvös gradiometric boundary value problem in the spectral domain we will need three systems of eigenfunctions, the complex scalar, spheroidal vector and shear tensor spherical harmonics. These functions form complete orthonormal systems of eigenfunctions for the incremental potential (scalar gravity anomaly), the incremental vectorial gradient and tensorial shear combination of observables, respectively.

In the second step we will formulate and solve the overdetermined problem for gravity anomalies by the least squares method in the spectral domain. This method is discussed by van Gelderen and Rummel (2001). We will see that the solution of the overdetermined problem is not unique in the sense that the spectral weights of the two observable combinations can be chosen freely, leading to different set-ups of the problem. The simplest choice of equal and unit spectral weights will lead to a particularly simple and closed-form solution as we will see in the next Section.
Many slightly different definitions of the scalar spherical harmonics $Y_{\ell m}$ exist. These differ by the normalization and phase convention. We have chosen the following definition of complex scalar spherical harmonics (see for example Bethe, 1933)

$$Y_{\ell m}(\phi, \lambda) := \left[ \frac{(2\ell + 1)(\ell - m)!}{4\pi (\ell + m)!} \right]^{1/2} \bar{P}_\ell^m(\sin \phi) e^{im\lambda}.$$  \hspace{1cm} (23)

This definition differs by the factor $(-1)^m$ from the phase convention of Condon and Shorley (1935). The symmetry of spherical harmonics of negative order $m$ is

$$Y_{-\ell m}(\phi, \lambda) = (-1)^m Y_{\ell m}(\phi, \lambda).$$ \hspace{1cm} (24)

The overbar over a complex quantity denotes complex conjugation here and in the discussion that follows. The Legendre functions $P_\ell^m(\sin \phi)$ and Legendre polynomials $P_\ell(x)$ are defined as follows:

$$P_\ell^m(x) := (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$ \hspace{1cm} (25)

$$P_\ell(x) := \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell.$$ \hspace{1cm} (26)

The complex scalar spherical harmonics (23) form a complete and orthonormal system on the unit sphere $S^2$

$$\langle Y_{\ell_1 m_1}(\phi, \lambda) | Y_{\ell_2 m_2}(\phi, \lambda) \rangle = \int \int_{0 - \pi/2}^{2\pi \pi/2} Y_{\ell_1 m_1}(\phi, \lambda) \overline{Y}_{\ell_2 m_2}(\phi, \lambda) \cos \phi \, d\phi \, d\lambda = \delta^2_{\ell_1} \delta^m_{m_1}.$$ \hspace{1cm} (27)

where $\delta^i_j := 1$ when $i = j$ and zero otherwise. Here and throughout this paper, $\langle . \mid . \rangle$ denotes the inner product of an appropriate Hilbert space of complex functionals.

The incremental potential $\delta w$ and the scalar gravity anomaly $\Delta \gamma$ can be transformed into the spectral domain of complex scalar spherical harmonics by the following decompositions

$$\delta w(\phi, \lambda, r) = U_0 \sum_{\ell=1}^\infty \left( \frac{R}{r} \right)^{\ell+1} \sum_{m=-\ell}^\ell \delta w_{\ell m} Y_{\ell m}(\phi, \lambda)$$ \hspace{1cm} (28)

$$\Delta \gamma(\phi, \lambda, r) = \gamma_0 \sum_{\ell=2}^\infty \left( \frac{R}{r} \right)^{\ell+2} \sum_{m=-\ell}^\ell \Delta \gamma_{\ell m} Y_{\ell m}(\phi, \lambda)$$ \hspace{1cm} (29)

outside the sphere of convergence $S^2_R$ with $U_0 := GM/R$ and $\gamma_0 := U_1/R$ where $GM$ denotes the geocentric gravitational constant. The low-degree harmonics $\ell = 0, 1$ are removed from these expansions by subtracting the reference field and because of the singularity of the Stokes operator $\nabla_{st}$ at $\ell = 1$. The coefficients of the above decompositions are obtained by observing the orthonormality of scalar spherical harmonics

$$\delta w_{\ell m} = \frac{1}{U_0} \int \left( \frac{R}{r} \right)^{\ell+1} \langle \delta w | Y_{\ell m} \rangle,$$ \hspace{1cm} (30)

and

$$\Delta \gamma_{\ell m} = \frac{1}{\gamma_0} \int \left( \frac{R}{r} \right)^{\ell+2} \langle \Delta \gamma | Y_{\ell m} \rangle$$ \hspace{1cm} (31)

for the incremental potential and gravity anomaly, respectively.

We now introduce the basis functions of tangent vector-valued functions defined on the unit sphere $S^2$. Any vector-valued function $t: S^2 \mapsto T S^2$, which belongs to some suitable Hilbert space can be expanded into complex spheroidal vector spherical harmonics $S_{\ell m}(\phi, \lambda)$. These harmonics are defined by the action of the surface gradient operator $\nabla_{surf}$ on scalar spherical harmonics

$$\nabla_{surf} Y_{\ell m}(\phi, \lambda) := \sqrt{\left( \ell + 1 \right)! \ell} S_{\ell m}(\phi, \lambda).$$ \hspace{1cm} (32)

The complex spheroidal vector harmonics obey the following orthonormality

$$\langle S_{\ell_1 m_1}(\phi, \lambda) | S_{\ell_2 m_2}(\phi, \lambda) \rangle := \int \int_{0 - \pi/2}^{2\pi \pi/2} S_{\ell_1 m_1}(\phi, \lambda) \cdot \overline{S}_{\ell_2 m_2}(\phi, \lambda) \cos \phi \, d\phi \, d\lambda = \delta^2_{\ell_1} \delta^m_{m_1}$$ \hspace{1cm} (33)

with respect to the inner (dot) product of two vector harmonics. They have a simple connection with spin 1 spherical harmonics which will be utilized in Appendix A to derive the addition theorem of these spheroidal vector harmonics with scalar harmonics. This relation will be needed to derive the solution of the Eötvös problem in the spatial domain.

Since the incremental gradient observable $\delta \mathbf{G}^{(1)}$ is a vector in the local tangent space $T S^2_R$ to the sphere $S^2_R$, it can be decomposed into complex spheroidal vector spherical harmonics outside $S^2_R$ as

$$\delta \mathbf{G}^{(1)}(\phi, \lambda, r) = \Gamma_0 \sum_{\ell=1}^\infty \left( \frac{R}{r} \right)^{\ell+3} \sum_{m=-\ell}^\ell \delta \mathbf{G}^{(1)}_{\ell m} S_{\ell m}(\phi, \lambda),$$ \hspace{1cm} (34)

where $\Gamma_0 = \gamma_0/R$. Again, the low degree harmonic $S_{00}(\ell = 0)$ has been removed by the reference gravity gradient field. The spectral coefficients $\delta \mathbf{G}^{(1)}_{\ell m}$ are
suitable Hilbert space can be expanded into complex function spherical harmonics shear tensor harmonics fact that any symmetric and trace-free tensor-valued (curvature) combination of observations, and trace-free, it has the following decomposition theorem in the spectral domain, the tensor-valued shear tensor harmonics have a simple relationship details the reader is again referred to Appendix A.

If we evaluate the integral in the above expression we have to take the inner product of the two vectors $\delta \Gamma^{(1)} \cdot \mathbf{S}_{lm}$ as Eq. (33) shows. Since the covariant vector-valued differential operator $\nabla^{(1)}$ (Eq. 11) has also a radial derivative part, the total action of $\nabla^{(1)}$ on the solid spherical harmonics is

$$\nabla^{(1)} Y_{lm}(\phi, \lambda) = (\ell + 2) \sqrt{\frac{(\ell + 1)!}{(\ell - 1)!}} \mathbf{S}_{lm} R^{\ell+3}, \quad (36)$$

To complete the first step to set up the Eötvös problem in the spectral domain, the tensor-valued shear (curvature) combination of observations, $\delta \Gamma^{(2)}$ have to be represented by complex tensor-valued shear spherical harmonics $\mathbf{Z}_{lm}$. This follows from the fact that any symmetric and trace-free tensor-valued function $T$: $\mathbb{S}^2 \rightarrow T^2 \otimes T^2$, which belongs to some suitable Hilbert space can be expanded into complex shear tensor harmonics $\mathbf{Z}_{lm}$. These are defined as

$$\nabla^{(2)} Y_{lm}(\phi, \lambda) := \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} \mathbf{Z}_{lm}(\phi, \lambda). \quad (37)$$

They are also orthonormal on $\mathbb{S}^2$ with respect to the inner product of two tensors (denoted by the colon operator ‘:’) $\langle \mathbf{Z}_{l_1m_1}(\phi, \lambda) | \mathbf{Z}_{l_2m_2}(\phi, \lambda) \rangle$

$$:= \int_0^{2\pi} \int_0^{\pi/2} \mathbf{Z}_{l_1m_1}(\phi, \lambda) : \mathbf{Z}_{l_2m_2}(\phi, \lambda) \cos \phi \, d\phi \, d\lambda = \delta_{l_1, l_2} \delta_{m_1, m_2}, \quad (38)$$

where the inner product of two tensors (contraction) is the sum over all elements of this product,

$$\mathbf{A} : \mathbf{B} := \sum_i \sum_j A_{ij} B_{ij}. \quad$$

Similarly to the vector spherical harmonics $\mathbf{S}_{lm}$, these shear tensor harmonics have a simple relationship with spin 2 spherical harmonics and an addition theorem with complex scalar spherical harmonics. For details the reader is again referred to Appendix A.

Since the curvature combination $\delta \Gamma^{(2)}$ is symmetric and trace-free, it has the following decomposition outside $\mathbb{S}^2$.

$$\delta \Gamma^{(2)}(\phi, \lambda, r) = \Gamma_0 \sum_{\ell=1}^{\infty} \left( \frac{r}{R} \right)^{\ell+3} \sum_{m=-\ell}^{\ell} \delta \Gamma^{(2)}_{lm} Z_{lm}(\phi, \lambda). \quad (39)$$

The spectral coefficients $\delta \Gamma^{(2)}_{lm}$ will follow from the orthonormality property Eq. (38) of $\mathbf{Z}_{lm}$.

$$\delta \Gamma^{(2)}_{lm} = \frac{1}{R} \left( \frac{r}{R} \right)^{\ell+3} \langle \delta \Gamma^{(2)} | \mathbf{Z}_{lm} \rangle, \quad (40)$$

where again the tensorial inner product $\langle \delta \Gamma^{(2)} : \mathbf{Z}_{lm} \rangle$ have to be formed. We summarize the first step of this Section in Figure 4. Here we can see that three systems of basis functions, $Y_{lm}$, $\mathbf{S}_{lm}$ and $\mathbf{Z}_{lm}$ are needed to transform the target $\Delta \mathbf{y}$ as well as the two incremental Eötvös observable combinations $\delta \Gamma^{(1)}$ and $\delta \Gamma^{(2)}$, respectively into the spectral domain of these basis functions. That is Eqs. (32) and (37) define the singular value decomposition (SVD) of the operators $\nabla_{surf}$ and $\nabla^{(2)}$, where the basis functions of the domain space of these operators are the complex scalar spherical harmonics $Y_{lm}$ and we have as basis functions in the range space of spheroidal vector spherical harmonics $\mathbf{S}_{lm}$ and shear tensor spherical harmonics $\mathbf{Z}_{lm}$, respectively (see Rummel, 1997).

We are now in the positon that we can set up and solve the Eötvös overdetermined gradiometric boundary value problem in the spectral domain. It is because we are able to represent the linear differential operators connecting the incremental vector and tensor observable combinations and the incremental potential and scalar gravity anomaly in the spectral domain. This connection is characterized by the spectral eigenvalues (singular values) of these spherical differential operators, the singular values acting as multipliers between spherical harmonic spectral coefficients. For vector- and tensor-valued second-order differential operators $\nabla^{(1)}$ and $\nabla^{(2)}$ these singular values have already been obtained by Eqs. (36) and (37). We will denote these singular values by

$$\lambda^{(1)}_\ell := (\ell + 2) \sqrt{\ell(\ell + 1)} \quad (41)$$

$$\lambda^{(2)}_\ell := \sqrt{(\ell - 1)(\ell + 1)(\ell + 2)}. \quad (42)$$

The action of the Stokes operator $\nabla_{St}$ on solid spherical harmonics is

$$\nabla_{St} \frac{Y_{lm}(\phi, \lambda)}{p^{\ell+1}} = (\ell - 1) \frac{Y_{lm}(\phi, \lambda)}{p^{\ell+2}}. \quad (43)$$

Therefore the following eigenvalue connections in the spectral domain can be established for each ob-
incremental potential

scalar gravity anomaly

vector-valued gradient observable

tensor-valued shear observable

The solution is

\[ \dot{\gamma}_{\ell m} = \frac{(\ell - 1)\lambda_{\ell}^{(1)} p_{\ell}^{(1)}}{(\lambda_{\ell}^{(1)})^2 p_{\ell}^{(1)} + (\lambda_{\ell}^{(2)})^2 p_{\ell}^{(2)}} \delta \Gamma_{\ell m}^{(1)} \]

\[ + \frac{(\ell - 1)\lambda_{\ell}^{(2)} p_{\ell}^{(2)}}{(\lambda_{\ell}^{(1)})^2 p_{\ell}^{(1)} + (\lambda_{\ell}^{(2)})^2 p_{\ell}^{(2)}} \delta \Gamma_{\ell m}^{(2)} \]  

Here we note that the low degree harmonics of \( \Delta \gamma_{\ell m} \) for \( \ell = 0, 1 \) (inadmissible harmonics) cannot be determined from the above solution. These, if required, must be determined separately. This feature is similar to the Stokes problem. Now the most simple choice of spectral weights is \( p_{\ell}^{(1)} = p_{\ell}^{(2)} = 1 \) for all \( \ell \). This leads to simple spectral multiplicative factors of \( \dot{g}_{\ell}^{(1)} \) and \( \dot{g}_{\ell}^{(2)} \) as follows

\[ \Delta \gamma_{\ell m} = \dot{g}_{\ell}^{(1)} \delta \Gamma_{\ell m}^{(1)} + \dot{g}_{\ell}^{(2)} \delta \Gamma_{\ell m}^{(2)} \]  

with

\[ \dot{g}_{\ell}^{(1)} = \frac{\ell - 1}{(2\ell + 1) \sqrt{\ell(\ell + 1)}} \]

\[ \dot{g}_{\ell}^{(2)} = \frac{(\ell - 1)^{3/2}}{(2\ell + 1) \sqrt{\ell(\ell + 1)(\ell + 2)}} \]  

We mention that this solution can easily be specialized to two uniquely determined gradiometric boundary value problems by setting the weight of one combination zero, that is either \( p_{\ell}^{(1)} = 0 \) or \( p_{\ell}^{(2)} = 0 \) for each degree \( \ell \). In this case we have the following factors for unit weights after substitution into Eq. (46)

\[ g_{\ell}^{(1)} = \frac{\sqrt{\ell - 1}}{\sqrt{\ell(\ell + 1)(\ell + 2)}} \]  

and

\[ g_{\ell}^{(2)} = \frac{(\ell - 1)}{(\ell + 2) \sqrt{\ell(\ell + 1)}} \]  

To conclude this Section, we summarize the solution of the overdetermined Eötvös gradiometric boundary value problem in the spectral domain in Figure 5.

4 Closed-form solution of the overde
determined gradiometric boundary value problem in the spatial domain by Green functions

In order to connect our target of the boundary value problem, that is gravity anomalies outside the reference sphere \( S_{R}^2 \), and the vector- and tensor-valued torsion balance observable combinations on the reference sphere \( S_{R}^2 \), we have to follow the scheme in Fig. 5. This way we will connect our target, \( \Delta \gamma \) in the spatial domain with observables \( \delta \Gamma^{(1)} \) and \( \delta \Gamma^{(2)} \) defined also in the spatial domain. The main point here is that certain vector- and tensor-valued Green functions, \( G_{\Delta \gamma}^{(1)} \) and \( G_{\Delta \gamma}^{(2)} \), will establish such a connection. By integrating the inner products of these
functions with the boundary data (observable combinations) over the reference sphere $S^2_R$, we will get the estimate $\Delta \hat{\gamma}$ of our target, the scalar-valued gravity anomalies on and outside of $S^2_R$.

The outline of this Section is summarized in the following six steps. We start with (i), scalar spherical harmonic decomposition of the gravity anomaly $\Delta \gamma$, then (ii) introduce the harmonic coefficients $\delta \Gamma_{\ell m}^{(1)}$ and $\delta \Gamma_{\ell m}^{(2)}$ into this expansion of $\Delta \hat{\gamma}$, (iii) replace these harmonic coefficients by their integral expressions over $S^2_R$, (iv) exchange series summation and integration, (v) apply addition theorems of vector and tensor harmonics to get the Legendre series of Green functions and finally (vi) produce closed-form expressions for the isotropic (rotationally invariant) part of these Green functions. We see that in Figure 5 these steps follow a right-to-left path from $\Delta \hat{\gamma}$ to $\delta \Gamma^{(1)}$ and $\delta \Gamma^{(2)}$ and thus establish the connections indicated by dashed lines directly in the spatial domain with Green functions $G_{\delta \gamma}^{(1)}$ and $G_{\delta \gamma}^{(2)}$.

**Step (i)** First we express our target, the scalar gravity anomaly $\Delta \gamma$ at the evaluation point $P$ outside of the reference sphere $P \in \mathbb{R}^3 - S^2_R$ by its spherical harmonic expansion with coefficients $\Delta \gamma_{\ell m}$ as it was done in Eq. (29)

$$\Delta \hat{\gamma}(\phi, \lambda, r) = \gamma_0 \sum_{\ell=1}^{\infty} \left(\frac{R}{r}\right)^{\ell+2} \sum_{m=-\ell}^{\ell} \Delta \gamma_{\ell m} Y_{\ell m}(\phi, \lambda). \quad (52)$$

**Step (ii)** The least squares solution of the overdetermined Eötvös gradiometric boundary value problem yielded an estimate of $\Delta \gamma_{\ell m}$ as a linear combination of spectral coefficients of the two observables in Eq. (47). We now introduce this estimate into Eq. (52) and thus we get two infinite series at the evaluation point $P(\phi, \lambda, r)$

$$\Delta \hat{\gamma}(\phi, \lambda, r) = \Delta \hat{\gamma}^{(1)}(\phi, \lambda, r) + \Delta \hat{\gamma}^{(2)}(\phi, \lambda, r), \quad (53)$$

where the sums are

$$\Delta \hat{\gamma}^{(1)}(\phi, \lambda, r) = \gamma_0 \sum_{\ell=2}^{\infty} \left(\frac{R}{r}\right)^{\ell+2} \sum_{m=-\ell}^{\ell} \Delta \hat{\gamma}_{\ell m}^{(1)} \delta \Gamma_{\ell m}^{(1)} Y_{\ell m}(\phi, \lambda) \quad (54)$$

and

$$\Delta \hat{\gamma}^{(2)}(\phi, \lambda, r) = \gamma_0 \sum_{\ell=2}^{\infty} \left(\frac{R}{r}\right)^{\ell+2} \sum_{m=-\ell}^{\ell} \Delta \hat{\gamma}_{\ell m}^{(2)} \delta \Gamma_{\ell m}^{(2)} Y_{\ell m}(\phi, \lambda). \quad (55)$$

**Step (iii)** To transform these expressions into the spatial domain we replace the harmonic coefficients of the observable combinations, $\delta \Gamma_{\ell m}^{(1)}$ and $\delta \Gamma_{\ell m}^{(2)}$ by their spatial domain expressions defined in Eqs. (35) and (40). These spatial domain integrals have to be evaluated at the source points $P^{\ast}(\phi^{\ast}, \lambda^{\ast}, r^{\ast})$ on $S^2_R$, where the boundary data $\delta \Gamma^{(1)}$ and $\delta \Gamma^{(2)}$ exist. Here now we make the distinction $r^{\ast} \neq R$ between the sphere on which we have the data and the sphere of convergence. Of course, the condition $r^{\ast} \geq R$ is required since the infinite series (34) and (39) have to be uniformly convergent.

The gradient and shear parts (54) and (55) become

$$\Delta \hat{\gamma}^{(1)}(\phi, \lambda, r) = r^{\ast} \sum_{\ell=2}^{\infty} \left(\frac{r^{\ast}}{r}\right)^{\ell+2} \Delta \hat{\gamma}_{\ell m}^{(1)} \delta \Gamma_{\ell m}^{(1)} Y_{\ell m}(\phi, \lambda) \quad (56)$$

and

$$\Delta \hat{\gamma}^{(2)}(\phi, \lambda, r) = \sum_{\ell=2}^{\infty} \left(\frac{r^{\ast}}{r}\right)^{\ell+2} \Delta \hat{\gamma}_{\ell m}^{(2)} \delta \Gamma_{\ell m}^{(2)} Y_{\ell m}(\phi, \lambda).$$

---

**Fig. 5:** Set-up and solution of the overdetermined Eötvös gradiometric boundary value problem by least squares method in the spectral domain, second step. The spatial domain solution by Green functions is indicated by dashed lines.
\[ \Delta \hat{s}^{(1)}(\phi, \lambda, r) = r' \sum_{\ell=0}^{\infty} \left( \frac{r'}{r} \right)^{\ell+2} \delta^{(2)}_{\ell} \times \sum_{m=-\ell}^{\ell} \langle \delta \Gamma_{lm}^{(2)}(\phi', \lambda', r') | Z_{lm}(\phi, \lambda) \rangle Y_{lm}(\phi, \lambda). \] (57)

**Step (iv)** If the above series expansions (56) and (57) are uniformly convergent, we are permitted to interchange summation and integration. These series are indeed uniformly convergent outside of the sphere \( S^2 \), that is where \( r \geq r' \) and hence we get the following equations

\[ \Delta \hat{s}^{(1)}(\phi, \lambda, r) = \frac{r'}{4\pi} \int_{S^2} \delta \Gamma^{(1)}(\phi', \lambda') \cdot G_{\Delta \lambda}^{(1)}(\phi, \lambda, r | \phi', \lambda', r') \, d\omega^*, \] (58)

\[ \Delta \hat{s}^{(2)}(\phi, \lambda, r) = \frac{r'}{4\pi} \int_{S^2} \delta \Gamma^{(2)}(\phi', \lambda') \cdot G_{\Delta \lambda}^{(2)}(\phi, \lambda, r | \phi', \lambda', r') \, d\omega^*, \] (59)

where \( d\omega^* = \cos \phi d\psi d\lambda \) denotes the solid angle element of the unit sphere \( S^2 \). It should also be noted that in Equations (58) and (59) we have to take the scalar-valued inner vector and tensor products (contractions) to get scalar-valued gravity anomalies. In this step the key parts of the integral expressions for the spatial domain solution were introduced, the **vector- and tensor-valued Green functions** \( G_{\Delta \lambda}^{(1)} \) and \( G_{\Delta \lambda}^{(2)} \). Their definitions are the following:

\[ G_{\Delta \lambda}^{(1)}(\phi, \lambda, r | \phi', \lambda', r') := 4\pi \sum_{\ell=0}^{\infty} \left( \frac{r'}{r} \right)^{\ell+2} \delta_{\ell}^{(1)} \sum_{m=-\ell}^{\ell} \bar{S}_{lm}(\phi', \lambda') Y_{lm}(\phi, \lambda). \] (60)

\[ G_{\Delta \lambda}^{(2)}(\phi, \lambda, r | \phi', \lambda', r') := 4\pi \sum_{\ell=0}^{\infty} \left( \frac{r'}{r} \right)^{\ell+2} \delta_{\ell}^{(2)} \sum_{m=-\ell}^{\ell} \bar{Z}_{lm}(\phi', \lambda') Y_{lm}(\phi, \lambda). \] (61)

These Green functions belong to the vector- and tensor-valued gradient and shear combinations of observables (incremental Eötvös observables).

**Step (v)** It is an interesting fact that the Green functions, which solve our gradimetric boundary value problem, can multiplicatively be decomposed into an azimuth-dependent vector- or tensor-valued part, and an azimuth-independent (isotropic) kernel. This decomposition is achieved most easily via the addition theorems of complex spheroidal vector and shear tensor harmonics. These theorems can be proved by taking into account the relation of these vector and tensor harmonics with spin-1 and spin-2 spherical harmonics. For details the reader is referred to Appendix A.

Fig. 6: Angular positions of source and evaluation points \( P(\phi', \lambda') \) and \( P(\phi, \lambda) \) projected to the unit sphere \( S^2 \). The azimuth \( \alpha^* \) towards the evaluation point \( P \) at the source point \( P^* \) is clockwise from North. The spherical distance \( \overline{PP^*} \) is denoted by \( \psi \)

\[ \sum_{m=-\ell}^{\ell} \bar{S}_{lm}(\phi', \lambda') Y_{lm}(\phi, \lambda) = \frac{2\ell + 1}{4\pi} \sqrt{(\ell+1)!} P^l(\cos \psi) f(\alpha^*) \] (62)

\[ \sum_{m=-\ell}^{\ell} \bar{Z}_{lm}(\phi', \lambda') Y_{lm}(\phi, \lambda) = \frac{2\ell + 1}{4\pi} \sqrt{(\ell+2)!} P^2(\cos \psi) F(\alpha^*). \] (63)

Here we denoted by \( f(\alpha^*) \) the

\[ f(\alpha^*) := \cos(\alpha^*) e^*_1 + \sin(\alpha^*) e^*_2 \] (64)

vector in the orthonormal \( \{ e^*_0, e^*_1 | P \} \) basis in the local tangent space \( T_P \cdot S^2 \), and \( F(\alpha^*) \) denotes the following STF shear tensor at \( P^* \), \( F(\alpha^*) \in T_{P^*} S^2 \otimes T_{P^*} S^2 \)

\[ F(\alpha^*) := \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 2\alpha^* & -\sin 2\alpha^* \\ -\sin 2\alpha^* & -\cos 2\alpha^* \end{pmatrix}. \] (65)

Now by observing these decompositions the Green functions will take the following forms

\[ G_{\Delta \lambda}^{(1)}(\phi, \lambda, r | \phi', \lambda', r') = G_{\Delta \lambda}^{(1)}(r, r', \psi) f(\alpha^*) \] (66)
The promised multiplicative decompositions of Green functions are established by the above two Eqs. (66), (67) into an azimuth-dependent vector-and tensor-valued part and a rotationally invariant isotropic part (kernel), one for each combination. These new isotropic kernels we will call later on the \( E\ddot{o}tv\ddot{o}s \) kernels and they have the following definitions

\[
G_{\Delta \gamma}^{(1)}(r, r^*, \psi) := \sum_{\ell=2}^{\infty} \left( \frac{r^*}{r} \right)^{\ell+2} (2\ell + 1) \sqrt{\frac{(\ell-1)!}{(\ell+2)!}} \beta_{1,\ell}(\psi) P_{\ell}^{(2)}(\cos \psi),
\]

\[
G_{\Delta \gamma}^{(2)}(r, r^*, \psi) := \sum_{\ell=2}^{\infty} \left( \frac{r^*}{r} \right)^{\ell+2} (2\ell + 1) \sqrt{\frac{(\ell-2)!}{(\ell+3)!}} \beta_{2,\ell}(\psi) P_{\ell}^{(2)}(\cos \psi). \tag{69}
\]

The \( E\ddot{o}tv\ddot{o}s \) kernels above are infinite series of Legendre functions of order 1 and 2. These functions will take the following forms if we substitute the definitions of the spectral coefficients \( \beta_{1,\ell}(\psi) \) and \( \beta_{2,\ell}(\psi) \) and the definition of Legendre functions \( P_{\ell}^{(2)}(\cos \psi) \) into them,

\[
G_{\Delta \gamma}^{(1)}(r, r^*, \psi) = -\frac{1}{\Delta \gamma} \hat{K}_{\Delta \gamma}^{(1)}(r, r^*, \psi)
:= -\frac{1}{\Delta \gamma} \sum_{\ell=2}^{\infty} \frac{\ell - 1}{(\ell + 1)} \left( \frac{r^*}{r} \right)^{\ell+2} P_{\ell}^{1}(\cos \psi), \tag{70}
\]

\[
G_{\Delta \gamma}^{(2)}(r, r^*, \psi) = \left( \frac{d}{d\rho} - \cot(\psi) \frac{d}{d\phi} \right) \hat{K}_{\Delta \gamma}^{(2)}(r, r^*, \psi)
:= \left( \frac{d}{d\rho} - \cot(\psi) \frac{d}{d\phi} \right) \sum_{\ell=2}^{\infty} \frac{\ell - 1}{(\ell + 1)(\ell + 2)} \left( \frac{r^*}{r} \right)^{\ell+2} P_{\ell}^{1}(\cos \psi). \tag{71}
\]

These expressions reveal that the \( E\ddot{o}tv\ddot{o}s \) kernels can be expressed by differentiating the corresponding series of Legendre polynomials, that is the two kernels \( \hat{K}_{\Delta \gamma}^{(1)}(r, r^*, \psi) \) and \( \hat{K}_{\Delta \gamma}^{(2)}(r, r^*, \psi) \), respectively.

Now we have the last step of this Section: we derive closed-form expressions of the \( E\ddot{o}tv\ddot{o}s \) kernels.

**Step (vi)** Since the derivation of the closed forms of the \( E\ddot{o}tv\ddot{o}s \) functions is lengthy, we only outline the derivation here. For details, the reader is to consult Appendix B. We start with partial fractional decompositions of \( \frac{\ell-1}{(\ell+1)} \) and \( \frac{\ell-1}{(\ell+1)(\ell+2)} \). Then we substitute closed-form expressions of known infinite sums of Legendre polynomials for each partial fraction and derive the closed-form expressions of \( \hat{K}_{\Delta \gamma}^{(1)}(r, r^*, \psi) \) and \( \hat{K}_{\Delta \gamma}^{(2)}(r, r^*, \psi) \). Finally the necessary derivations are taken according to Equations (70) and (71). The results are repeated here for reference:

\[
G_{\Delta \gamma}^{(1)}(r, r^*, \psi) = \sum_{\ell=2}^{\infty} \left( \frac{r^*}{r} \right)^{\ell+2} \frac{\ell - 1}{(\ell + 1)} P_{\ell}^{1}(\cos \psi)
= \frac{s^3 \sin \psi}{LN} (1 - L), \tag{72}
\]

\[
G_{\Delta \gamma}^{(2)}(r, r^*, \psi) = \sum_{\ell=2}^{\infty} \left( \frac{r^*}{r} \right)^{\ell+2} \frac{\ell - 1}{(\ell + 1)(\ell + 2)} P_{\ell}^{1}(\cos \psi)
= \frac{s^2 \sin^2 \psi}{2LN^2} [(3N - L - 3L^2 + 6L^2 + 6LN - 2L^2 - 3L) - 3s \cos \psi(N + L + L^2 - 2L^2 + 3L)], \tag{73}
\]

with the definitions

\[
s := \frac{(r^* / r)},
L := \sqrt{1 - 2s \cos \psi + s^2},
N := 1 - s \cos \psi + L.
\]

These functions are particularly simple when we set \( r^* = r \). Their graphs are shown in Figure 7 and we provide them for reference here:

![Fig. 7](image-url)

These functions are shown here as a function of the spherical distance \( \psi \) for the special case \( r^* = r \), that is when we have both the torsion balance observables and gravity anomalies on the same reference sphere \( S^2 \). Note the singularity of both functions at the origin \( \psi = 0 \).

\[
G_{\Delta \gamma}^{(1)}(\psi) = \frac{\cos \frac{\psi}{2} \left( 1 - \sin \frac{\psi}{2} \right)}{\sin \frac{\psi}{2} (1 + \sin \frac{\psi}{2})} \tag{74}
\]
\[ G_{\Delta y}^{(2)}(\psi) = \frac{(1 - \sin \frac{\psi}{2})^2}{2 \sin \frac{\psi}{2} (1 + \sin \frac{\psi}{2})} \]  

(75)

If we would like to transform the torsion balance observables into gravity anomalies, both located on the reference sphere \( S^2_R \), we will need these Green functions in Eqs. (74) and (75).

Now we take note of an important feature of these functions. They behave like the Stokes function \( S(\psi) \) near the origin \( \psi = 0 \). Why can we state this? It is well known that the Stokes function has the following approximation near the origin:

\[ S(\psi) \approx \frac{1}{\sin \frac{\psi}{2}} \]

The Green functions, Eqs. (74) and (75) of the Eötvös gradiometric boundary value problem, both have the following approximation at \( \psi = 0 \):

\[ G_{\Delta y}^{(1)}(\psi) = G_{\Delta y}^{(2)}(\psi) \approx \frac{1}{2 \sin \frac{\psi}{2}} \]

Therefore, since we have an overdetermined problem, the sums of these kernels we need according to Eq. (53), we can state that their sum behaves exactly like the Stokes function very close to the origin \( \psi = 0 \). This means that the same kind of singularity arises in the Eötvös gradiometric boundary value problem for gravity anomaly as its target as in the Stokes boundary value problem.

\[ G_{\Delta w}^{(1)}(\psi, \lambda) = \frac{1}{\cos \frac{\psi}{2}} - \tan \frac{\psi}{2} (1 + \cos^2 \frac{\psi}{2}) \]

\[ G_{\Delta w}^{(2)}(\psi, \lambda) = \frac{1}{1 + \sin \frac{\psi}{2}} - \frac{1}{2} \]

Therefore, since we have an overdetermined problem, the sums of these kernels we need according to Eq. (53), we can state that their sum behaves exactly like the Stokes function very close to the origin \( \psi = 0 \). This means that the same kind of singularity arises in the Eötvös gradiometric boundary value problem for gravity anomaly as its target as in the Stokes boundary value problem.

\[ \Delta \hat{\phi}^{(1)}(\phi, \lambda, r) = \frac{R}{4\pi} \int_{S^2_R} G_{\Delta y}^{(1)}(r, R, \psi) \times [\delta \Gamma_{13}(\phi^*, \lambda^*) \cos \alpha^* + \delta \Gamma_{23}(\phi^*, \lambda^*) \sin \alpha^*] d\omega^* \]

(78)

\[ \Delta \hat{\phi}^{(2)}(\phi, \lambda, r) = \frac{R}{4\pi} \int_{S^2_R} G_{\Delta y}^{(2)}(r, R, \psi) \times [\delta \Gamma_{13}(\phi^*, \lambda^*) \cos 2\alpha^* + \delta \Gamma_{12}(\phi^*, \lambda^*) \sin 2\alpha^*] d\omega^* \]

(79)

These expressions are called the \textit{Eötvös integrals}, and the estimate for scalar gravity anomalies from
the Eötvös observables is the sum of the above two contributions. It is interesting to note that Vassiliou (1986) has derived in planar approximation an expression very similar to the Eq. (78) between $\delta \Gamma_3$ and $\delta \Gamma_{13}$, $\delta \Gamma_{23}$. We find also identical expressions in van Gelderen and Rummel (2001), except for a sign difference in the $\sin 2\alpha^*$ term. The cause of this sign difference is unknown to us.

5 Truncation characteristics of isotropic Green functions in the Eötvös overdetermined gradiometric boundary value problem

The integration of torsion balance observables for the computation of gravity anomalies should cover the entire surface of the Earth. Nevertheless, if the integration is executed not over the whole surface of the reference sphere $S^2_R$ but only up to a spherical distance $\psi_0$, the evaluation of the truncation errors, or at least an estimation of the effect of neglecting the remote zone is required. This situation follows directly from the restricted spatial domain of measurements, which measurements are considered to be known only inside a spherical cap $S_0$ of radius $\psi_0$.

In this Section our main target is to examine in detail the truncation characteristics of the integral expressions (Eötvös integrals) (78) and (79). To reach this goal, first we transform the truncation error into the spectral domain. Because of the isotropy of the truncation, this transformation can be achieved via infinite series of Legendre functions of order 1 and 2. In the second part of this Section the coefficients of these Legendre series (truncation coefficients) will be expressed as integrals with Legendre polynomials, and we will show in detail how these coefficients can be evaluated. Thus we will give a computational "recipe" of truncation coefficients of the Eötvös kernels (both for gravity anomaly and incremental potential). In the final part of this Section we will see a very interesting difference between the truncation characteristics of the isotropic Green functions

Fig. 9: Comparison of a two-step versus one-step integration process of torsion balance observables. In the upper subfigure (spatial domain) the boundedness of Green functions at $\psi = 0$ spherical distance is indicated. Notice the difference in this respect between the two integration processes. The lower subfigure shows the same situation in the spectral domain. Here there is an exact correspondence between the two approaches. We indicate the spectral coefficients of observables as well as the spectral eigenvalues of Green functions.
(Eötvös kernels), which solve the boundary value problem for gravity versus incremental potential.

We start our examination with the definition of the truncation error of the Eötvös integrals in the spatial domain,

$$
\Delta \gamma_t(\phi, \lambda, r) := \Delta \gamma_t^{(1)}(\phi, \lambda, r) + \Delta \gamma_t^{(2)}(\phi, \lambda, r)
$$

where

$$
\Delta \gamma_t^{(1)}(\phi, \lambda, r) := \frac{R}{4\pi} \int_{S^2 - S_0} G_{\Delta \gamma}^{(1)}(r, R, \psi) \delta \Gamma^{(1)}(\phi^*, \lambda^*) \cdot f(\alpha^*) \mathrm{d}\omega^*
$$

and

$$
\Delta \gamma_t^{(2)}(\phi, \lambda, r) := \frac{R}{4\pi} \int_{S^2 - S_0} G_{\Delta \gamma}^{(2)}(r, R, \psi) \delta \Gamma^{(2)}(\phi^*, \lambda^*) \cdot F(\alpha^*) \mathrm{d}\omega^*.
$$

Here the lower index $t$ stands for the truncation variable $t = \cos \psi_0$. If now we introduce the error kernels (Green functions)

$$
\overline{G}_{\Delta \gamma}^{(i)}(r, R, \psi) := \begin{cases}
0 & \text{if } \psi \leq \psi_0 \\
G_{\Delta \gamma}^{(i)} & \text{if } \psi > \psi_0,
\end{cases}
$$

then each integral (81) and (82) can be extended over the whole sphere $S^2$ instead of the $S^2 - S_0$ (remote zone), by substituting the error kernels above instead of the original Green functions $G_{\Delta \gamma}^{(i)}$, $i \in \{1, 2\}$. (Since the Green functions are real-valued, no confusion with the notation of complex conjugate should arise from the overbar notation of the error kernels.)

The next step of the derivation is to introduce the spectral forms of isotropic error kernels. From here and onwards we make the restriction $R = r$, that is we make the error kernels independent of the radial variable. With this assumption the error kernels have the following expansions into Legendre function series

$$
\overline{G}_{\Delta \gamma}^{(i)}(\psi) := \sum_{\ell = 2}^{\infty} \frac{2\ell + 1}{2} \sqrt{\frac{(\ell - i)!}{(\ell + i)!}} \bar{Q}^{(i)}_\ell(\cos \psi),
$$

$$
\quad i \in \{1, 2\}.
$$

As we will prove now, the spectral form of the truncation error for each observable combination with index $i \in \{1, 2\}$ is the following:

$$
\Delta \gamma_t^{(i)}(\phi, \lambda) = \frac{R}{2} \sum_{\ell = 2}^{\infty} \bar{Q}^{(i)}_\ell \Gamma^{(i)}_\ell(\phi, \lambda),
$$

where $\Gamma^{(i)}_\ell$ denotes the $\ell$-th degree harmonic of the observable combination $i$. This has the following definition,

$$
\Gamma^{(i)}_\ell(\phi, \lambda) := \Gamma_0 \sum_{m = -\ell}^\ell \delta \Gamma^{(i)}_{\ell m} Y_{\ell m}(\phi, \lambda).
$$

We prove this statement for the vectorial gradient observable combination $\delta \Gamma^{(1)}$ only ($i = 1$), since the proof for the tensor-valued shear combination is completely analogous. The idea of the proof is to transform Eq. (85), by substituting Eq. (86) into it, to the definition Eq. (81).

To start the proof, we substitute the definition of the harmonic coefficient $\delta \Gamma^{(1)}_{\ell m}$, Eq. (35) with $R = r$ into the definition of the $\ell$-th degree harmonic in Eq. (86). This yields the following expression after exchanging summation and integration

$$
\Gamma^{(i)}_\ell(\phi, \lambda) = \int_{S^2} \delta \Gamma^{(i)} \cdot \sum_{m = -\ell}^\ell \bar{S}_{\ell m}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \mathrm{d}\omega^*.
$$

Next, by virtue of the addition theorem of spheroidal vector harmonics, Eq. (62) we have

$$
\Gamma^{(i)}_\ell(\phi, \lambda) = \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!}} \int_{S^2} \delta \Gamma^{(i)} \cdot f(\alpha^*) P^i_\ell(\cos \psi) \mathrm{d}\omega^*.
$$

Finally we substitute this expression of the $\ell$-th degree harmonic into Eq. (85). Since this Legendre series expansion is uniformly convergent we are permitted to interchange the order of summation with integration, and finally we get the following equation of the truncation error as

$$
\gamma_t^{(i)}(\phi, \lambda) = \frac{R}{2} \sum_{\ell = 2}^{\infty} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!}} \times Q^{(i)}_\ell P^i_\ell(\cos \psi) \mathrm{d}\omega^*.
$$

But this expression is indeed the spatial domain definition of the truncation error of the gradient combination when we take into account the spectral form of the truncation error kernel $\overline{G}_{\Delta \gamma}^{(i)}(\psi)$ in Eq. (84). Thus the proof is complete. Therefore Eq. (85) is the desired spectral decomposition of the truncation error for each incremental Eötvös observable combination with the truncation coefficients in Eq. (84).

Now we turn our attention to the second part of this Section. To start with, we give explicit integral
expressions of the truncation error coefficients of the Eötvös kernels \( G^{(1)}_{\Delta \gamma}(r, r') \). These come from the Legendre series, Eq. (84) by observing the following orthonormality relation

\[
\int_0^\pi [P^i_\ell(\cos \psi)]^2 \sin \psi d\psi = \frac{2(\ell + i)!}{(2\ell + 1)(\ell - i)!}, \quad i \in \{1, 2\}
\]

(90)
of the Legendre functions \( P^i_\ell \), yielding

\[
Q^{(i)}_\ell(\psi_0) = \frac{(\ell - i)!}{(\ell + i)!} \int_0^\pi G^{(i)}_{\Delta \gamma}(\psi) P^i_\ell(\cos \psi) \sin \psi d\psi
\]

(91)
with \( i \in \{1, 2\} \). This integral expresses the truncation coefficients \( Q^{(i)}_\ell \) in terms of integrals of the Legendre functions \( P^i_\ell(\cos \psi) \), \( P^2_\ell(\cos \psi) \) and the Eötvös kernels.

Paul (1973) has developed a compact method of computing the truncation error coefficients of the Stokes function \( S(\psi) \).

\[
Q^{(i)}_\ell(\psi_0) = \int_0^\pi S(\psi) P^i_\ell(\cos \psi) \sin \psi d\psi.
\]

(92)
His evaluation method utilizes the fact that the above integrals for \( Q^{(i)}_\ell \) contain the Legendre polynomials. But unfortunately, our Eqs. (91) contain first and second order Legendre functions instead. In order to derive an evaluation "recipe" for \( Q^{(i)}_\ell \), we have the idea of transforming the integrals into other expressions which contain the Legendre polynomials instead of Legendre functions. After this transformation they will have a form suitable to an adapted version of Paul’s method to evaluate them in a recursive scheme. This idea of transformation is not new. Hagiwara (1973) has derived in a similar manner the connection between the truncation coefficients of the Stokes and Vening-Meinesz formulas. We will only extend this connection up to the second-order derivatives.

As we will show next, the evaluation of the truncation error coefficients \( Q^{(i)}_\ell \) can be reduced to the evaluation of the following coefficients

\[
K^{(i)}_\ell(\psi_0) = \int_0^\pi \hat{K}^{(i)}_{\Delta \gamma}(\psi) P^i_\ell(\cos \psi) \sin \psi d\psi,
\]

(93)
which are the truncation coefficients of the kernels \( \hat{K}^{(i)}_{\Delta \gamma}(r, r', \psi) \) in Eqs. (70) and (71). The Eötvös kernels can be derived from these kernels by one- or twofold differentiations. First we derive this relation for the gradient combination \( s = 1 \). We substitute \( -\frac{d}{d\psi} \hat{K}^{(i)}_{\Delta \gamma}(\psi) \) into Eq. (91) for \( G^{(1)}_{\Delta \gamma}(\psi) \). Then we partially integrate the resulting expression,

\[
\int_0^\pi \frac{d\hat{K}^{(i)}_{\Delta \gamma}}{d\psi} P^i_\ell(\cos \psi) \sin \psi d\psi = \hat{K}^{(i)}_{\Delta \gamma}(\psi) P^i_\ell(\cos \psi) \sin \psi \bigg|_0^\pi - \int_0^\pi \hat{K}^{(i)}_{\Delta \gamma}(\psi) \frac{d}{d\psi} [P^i_\ell(\cos \psi) \sin \psi] d\psi.
\]

In the second term we can compute the derivative of \( P^i_\ell(\cos \psi) \sin \psi \) by observing the following relation (see Hagiwara, 1973, Eq. 36)

\[
\frac{d}{d\psi} [P^i_\ell(\cos \psi) \sin \psi] = \ell(\ell + 1) P^i_\ell(\cos \psi) \sin \psi.
\]

(95)
Thus we get the desired relation between \( Q^{(1)}_\ell \) and \( K^{(1)}_\ell \) as

\[
Q^{(1)}_\ell(\psi_0) = \int_0^\pi S(\psi) P^1_\ell(\cos \psi) \sin \psi d\psi = \hat{K}^{(1)}_{\Delta \gamma}(\psi_0) P^1_\ell(\cos \psi_0) \sin \psi_0 + \ell(\ell + 1) K^{(1)}_\ell(\psi_0).
\]

(96)
The derivation of the corresponding relation for the truncation coefficients of the shear combination is similar but more lengthy. First we write the integral in Eq. (91) as

\[
\int_0^\pi G^{(2)}_{\Delta \gamma}(\psi) P^2_\ell(\cos \psi) \sin \psi d\psi = \int_0^\pi \left( \frac{d^2}{d\psi^2} - \cot(\psi) \frac{d}{d\psi} \right) \hat{K}^{(2)}_{\Delta \gamma}(\psi) P^2_\ell(\cos \psi) \sin \psi d\psi.
\]

(97)
Then we evaluate it by successive partial integrations, but first we simplify the notation with the definitions \( x := \cos \psi, t := \cos \psi_0 \) and \( K := \hat{K}^{(2)}_{\Delta \gamma} \).
yields
\[
\int_{-1}^{1} \frac{d^2}{dx^2} (1 - x^2) P_\ell^2(x) \, dx = \left. \frac{dK}{dx} (1 - x^2) P_\ell^2(x) \right|_{x=-1}^{x=1} - \int_{-1}^{1} \frac{dK}{dx} \, [(1 - x^2) P_\ell^2(x)] \, dx = \left. \frac{dK}{dx} \right|_{x=t} (1 - t^2) P_\ell^2(t) - K(t) \, \left. \frac{d}{dx} [(1 - x^2) P_\ell^2(x)] \right|_{x=-1}^{x=t} + \int_{-1}^{t} K(x) \, \frac{d^2}{dx^2} [(1 - x^2) P_\ell^2(x)] \, dx.
\]

(98)

The following relation
\[
\frac{d^2}{dx^2} \left[ (1 - x^2) \frac{d^2 P_\ell(x)}{dx^2} \right] = \frac{(\ell + 2)!}{(\ell - 2)!} P_\ell(x)
\]

(99)
can be proved easily from the differential equation of Legendre polynomials $P_\ell(x)$, but because the derivation is somewhat lengthy and rather technical we omit here. Observing this relation, Eq. (99) in the last term of Eq. (98), we get again to the desired connection between $Q_\ell^{(2)}$ and $K_\ell^{(2)}$:
\[
Q_\ell^{(2)}(\psi_0) = \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left( \frac{dK^{(2)}_\ell(\psi)}{d\psi} \right)_{\psi=\psi_0} P_\ell^2(\cos \psi_0) \sin \psi_0 + \frac{K^{(2)}_\ell(\psi_0)}{2} \cos \psi_0 P_\ell^2(\cos \psi_0) + \frac{dP_\ell^2(\cos \psi)}{d\psi} \bigg|_{\psi=\psi_0} \sin \psi_0 + \frac{(\ell + 2)!}{(\ell - 2)!} K_\ell^{(2)}(\psi_0).
\]

(100)

To compute the truncation coefficients $Q_\ell^{(1)}(\psi_0)$ and $Q_\ell^{(2)}(\psi_0)$ from Eqs. (96) and (100) respectively, we need the truncation coefficients $K_\ell^{(2)}(\psi_0)$ (see Eq. (93)) as well as the function values $K^{(1)}_\ell(\psi)$ at $\psi = \psi_0$ and its derivatives. This accomplishment finishes the second part of this Section. For reference, we summarize the closed-form expressions of the kernels and derivative kernels for the Eötvös gradiometric boundary value problem of gravity anomalies as well as incremental potential in Table 1.

Now in the main part of this section we will propose an evaluation method of the truncation error coefficients defined by the Eq. (93). As we have noted earlier in this Section, this integral is very similar to the Stokes integral in an important respect, that is we find the Legendre polynomials in both Eq. (93) and Eq. (92). Hence, the idea of Paul (1973) can easily be adapted to the evaluation of these truncation error coefficients with a minor addition, although the final expressions are of course completely different. For completeness the truncation coefficient computation formulas of all the four Eötvös kernels in Table 1 are included here, both for gravity anomaly $\Delta \gamma$ and for incremental potential $\delta w$ as target. Since we have altogether four different sets of computation formulas, and the detailed derivations are not hard but tedious, we only summarize here the main steps of it and some basic definitions. For the ready-to-compute specific formulas the interested reader should consult the Equations of Tables 2 and 3. We make the remark that we have intentionally kept our notation here similar to that of Paul (1973) for easy reference and to help the reader to follow our derivation more easily.

We start with a general definition of the truncation coefficients $K_\ell(t)$ in Eq. (93), where the abbreviations $t := \cos \psi_0$ and $z := \cos \psi$ are used. This definition is as follows:
\[
K_\ell(t) := \int_{-1}^{t} K(z) P_\ell(z) \, dz,
\]

(101)

where the $K(z)$ kernel function can be represented in any particular case in its spectral form by the following Legendre series
\[
K(z) = \sum_{k=2}^{\infty} g(k) P_k(z).
\]

(102)

The spectral coefficients $g(k)$ of a particular Green function for each observable combination $i \ (i \in (1, 2))$ and target (gravity anomaly $\Delta \gamma$, incremental potential $\delta w$) are any of the following four ones (see Fig. 9):

\[
\begin{align*}
g_\ell^{(1)}(k) &= \frac{k - 1}{k(k + 1)} \\
g_\ell^{(1)}(k) &= \frac{1}{k(k + 1)} \\
g_\ell^{(2)}(k) &= \frac{k - 1}{k(k + 1)(k + 2)} \\
g_\ell^{(2)}(k) &= \frac{1}{k(k + 1)(k + 2)}
\end{align*}
\]

(103)

Each upper index of $g(k)$ denotes an observable combination and each lower index a target functional.
Table 1: Closed-form expressions of different isotropic Green functions (Eötvös kernels) and derivative functions of the Eötvös gradiometric boundary value problems for gravity anomaly and incremental potential. All kernels are restricted to the sphere $r = r' = R$, that is independent of the radial variable. We have used the abbreviations $s := \sin(\psi/2)$, $c := \cos(\psi/2)$.

<table>
<thead>
<tr>
<th>Observable combination</th>
<th>derivative</th>
<th>target: gravity anomaly $\Delta \gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \Gamma^{(1)}$</td>
<td>(gradient)</td>
<td>$-2 + 2 \ln\left(1 + \frac{1}{s}\right) + \ln(s + s^2)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d}{d\psi}$</td>
<td>$\frac{c(1 - s)}{s(1 + s)}$</td>
</tr>
<tr>
<td>$\delta \Gamma^{(2)}$</td>
<td>(shear)</td>
<td>$\frac{1}{4} - 3s + \left(\frac{1}{2} + 3s^2\right) \ln\left(1 + \frac{1}{s}\right) + \frac{1}{2} \ln(s + s^2)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d}{d\psi}$</td>
<td>$sc\left[3 \ln\left(1 + \frac{1}{s}\right) - \frac{3s + 1}{s(s + 1)}\right]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Observable combination</th>
<th>derivative</th>
<th>target: incremental potential $\delta \omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \Gamma^{(1)}$</td>
<td>(gradient)</td>
<td>$\frac{1}{2} + s^2 - \ln(s + s^2) - \ln\left(1 + \frac{1}{s}\right)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d}{d\psi}$</td>
<td>$\frac{1 - s(2 - s^2)}{c}$</td>
</tr>
<tr>
<td>$\delta \Gamma^{(2)}$</td>
<td>(shear)</td>
<td>$\frac{1}{12} + \frac{1}{3} s^2 + s - \frac{1}{2} \ln(s + s^2) + \left(\frac{1}{2} - s^2\right) \ln\left(1 + \frac{1}{s}\right)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d}{d\psi}$</td>
<td>$sc\left[\frac{1}{3} + \frac{1}{1 + s} - \ln\left(1 + \frac{1}{s}\right)\right]$</td>
</tr>
</tbody>
</table>

Next we substitute Eq. (102) into Eq. (101) and interchange the order of summation with integration: and these yield the following expression for the truncation coefficients Eq. (104):

$$K_r(t) = \sum_{k=\ell}^{\infty} g(k) \int_{-1}^{1} P_r(z) P_k(z) dz + g(\ell) \int_{-1}^{1} [P_\ell(z)]^2 dz. \quad (104)$$

$$K_r(t) = \sum_{k=\ell}^{\infty} g(k) R_{\ell,k}(t) + g(\ell) R_{\ell,\ell}(t). \quad (107)$$

To further transform this equation, we define the following coefficients

$$R_{\ell,k}(t) := \int_{-1}^{1} P_r(z) P_k(z) dz \quad (105)$$

and

$$R_{\ell,\ell} := \int_{-1}^{1} [P_\ell(z)]^2 dz. \quad (106)$$

In order to evaluate the infinite sum in the first term, Paul (1973, Eq. 5) shows that the following relation holds:

$$R_{\ell,k}(t) = \left[\frac{\ell + 1}{2} \right] P_\ell(t)[P_{\ell+1}(t) - P_{\ell-1}(t)] - \frac{k+1}{2k+1} P_\ell(t)[P_{k+1}(t) - P_{k-1}(t)] \quad (108)$$

When we substitute this expression of $R_{\ell,k}(t)$ into Eq. (107) and a particular explicit form of $g(k)$, from Eq.
(103) we are able to do the partial fractional decomposition of the resulting expressions with respect to \( k \) into terms like
\[
\frac{1}{k \pm \nu} \quad \text{and} \quad \frac{1}{2k-1}, \quad \frac{1}{2k+3}.
\]
In his derivation Paul used the explicit form of the \( g(k) \) coefficients of the Stokes function, \( g(k) = (2\ell + 1)/(\ell - 1) \). He needed to evaluate only infinite sums of terms like \( 1/(k + \nu) \). He showed that each partial fractional of the form \( 1/(k \pm \nu) \) can be summed up with respect to \( k \) as the special case of either the infinite sum
\[
U_r(t) := \sum_{k=0}^{\infty} \frac{P_k(t)}{k - \ell + 1}, \quad l \in \{0, 1, 2, \ldots\} \quad (109)
\]
or
\[
U_r(t) := \sum_{k=0}^{\infty} \frac{P_k(t)}{k + \ell + 1}, \quad l \in \{0, 1, 2, \ldots\} \quad (110)
\]
These infinite sums can be recursively computed by the method of Paul (1973). His formulas are summarized in Table 2 for reference. The spectral coefficients \( g(k) \), Eqs. (103) introduce the additional infinite sum of terms like \( 1/(2k - 1) \) and \( 1/(2k + 3) \), but fortunately these partial fractions in each of our four cases can be transformed into the following infinite sum
\[
W(t) := \sum_{k=2}^{\infty} \frac{P_{k+1}(t) - P_{k-1}(t)}{2k + 1} \quad (111)
\]
This infinite sum can be easily computed by observing the following relation
\[
\frac{P_{k+1}(t) - P_{k-1}(t)}{2k + 1} = P_k(t)
\]
and the known sum
\[
\sum_{k=2}^{\infty} P_k(t) = -1 - t + \frac{1}{\sqrt{2 - 2t}}.
\]
Then it follows after integration and taking into account the fact that \( P_1(t) = 1 \) for all \( k \in \{0, 1, 2, \ldots\} \), that the result is the following closed expression of the sum \( W(t) \):
\[
W(t) = \frac{3}{2} - t - \frac{t^2}{2} - \sqrt{2 - 2t}. \quad (112)
\]
This sum was not needed in Paul’s scheme for the truncation coefficient computation of the Stokes integral, but as we have just seen it is needed for the truncation computation of Eötvös integrals.

Now we are in the position to express the first term, the infinite sum in Eq. (107) as a combination of the known sums \( U_r(t), V_r(t) \) and \( W(t) \). The second term in this Equation contains the factor \( K_{\ell}(t) \), which also enjoys a recursive evaluation scheme discussed in Paul (1973). This means that each of the four set of truncation coefficients \( K_{\ell}(t) \), corresponding to a set of spectral coefficients in Eq. (103) can be evaluated now by a recursive computational scheme. Although the partial fractional decompositions and the corresponding sums are tedious to evaluate, the process is straightforward and leads to the formulas of Table 3.

We note that these truncation coefficients have yet to be transformed either by Eq. (96) or Eq. (100) to the truncation coefficients of the Eötvös kernels.

To check our formulas in Table 3 numerically we have developed Fortran 90 routines for the evaluation of truncation error of Eötvös kernels. A good method of check is to approximate the kernel \( K(z) \) numerically by evaluating its Legendre expansion according to Eq. (102) up to a certain maximum degree \( k_{\text{max}} \), and truncation radius \( t = \cos \psi_0 \), and check the convergence of the series to the analytically defined error kernel \( \overline{K}(z) \). The numerical checks confirmed that these series indeed sum approximately up to the required analytical error kernels in each of the four cases in Table 3. As a graphical illustration of convergence we present Figure 10. Here an error kernel is synthesized for the gradient observable Eötvös kernel (target: gravity anomalies). The maximum degree of the expansion was relatively low, \( k_{\text{max}} = 36 \). One should notice nevertheless the clear convergence of the synthesized error kernel to the true analytical kernel.

We have now arrived to the last part of this Section and since we know how to quantize the effect of truncation on each of the four Eötvös kernels, we can make a clear point. This is the following.

If we have gravity gradient (torsion balance) data available over a limited area only, it is more advantageous to transform them to geoid undulations in a two-step integration process. That is, compute gravity anomalies first and then geoid undulations (incremental potential). This is true at least when the truncation error of the integration is considered.

We will illustrate our point by computing the average truncation error of gravity anomalies and geoid undulations as a function of truncation cap size (radius) for a realistic gravity field model of the Earth.

It can be shown (see Tóth et al., 2002) that the average square of the truncation error on the gravity field quantity \( f \) (either gravity anomaly \( \Delta \gamma \) or geoid undulation \( \mathcal{N} \) over the spherical harmonic de-
Table 2: Truncation error coefficient computation formulas for the Eötvös kernels (isotropic Green functions) — Part 1. The method is similar to that of Paul (1973). For more explanation see text. The following abbreviations are used: $s := \sin(\psi_0/2)$, $t := \cos(\psi_0)$, where $\psi_0$ is the truncation cap radius.

Computation scheme of $R_{\ell,2}(t)$ (see Paul, 1973, Eq. 6)

$$R_{0,0}(t) = t + 1$$
$$R_{1,2}(t) = (t^3 + 1)/3$$
$$R_{\ell,2}(t) = \frac{\ell(2\ell - 1)}{2\ell + 1} R_{\ell-1,2}(t) - \frac{\ell - 1}{\ell - 1} R_{\ell,2}(t) + \frac{2\ell - 1}{2\ell + 1} R_{\ell-1,1}(t)$$

(113) (114) (115)

Computation scheme of $R_{\ell,2}(t)$ ($\ell \neq k$) (see Paul, 1973, Eq. 5)

$$R_{\ell,2}(t) = \frac{1}{(\ell - k)(\ell + k + 1)} \left( \frac{\ell(\ell + 1)}{2\ell + 1} P_{\ell}(t) [P_{\ell + 1}(t) - P_{\ell - 1}(t)] - \frac{k(k + 1)}{2k + 1} P_{\ell}(t) [P_{\ell + 1}(t) - P_{\ell - 1}(t)] \right)$$

(116)

Recursive computation of $U_{\ell}(t)$ (see Paul, 1973, Eqs. 20, 21)

$$U_{0}(t) = \ln \left( 1 + \frac{2}{\sqrt{2 - 2t}} \right), \quad t \neq 1 \quad U_0(t) = 0 \quad t = 1$$
$$U_{1}(t) = \ln \frac{2}{1 - t + \sqrt{2 - 2t}}, \quad t \neq 1 \quad U_1(t) = 0 \quad t = 1$$
$$U_{\ell}(t) = \frac{1}{\ell - 1} \left( (2\ell - 3)tU_{\ell-1}(t) - (\ell - 2)U_{\ell-2}(t) - \sqrt{2 - 2t} + \frac{P_{\ell-3}(t) - P_{\ell-1}(t)}{2\ell - 3} \right) \quad \ell = 2, 3, \ldots$$

(117) (118) (119)

Recursive computation of $V_{\ell}(t)$ (see Paul, 1973, Eqs. 23, 24)

$$V_{0}(t) = U_0(t)$$
$$V_{1}(t) = tV_{0}(t) + \sqrt{2 - 2t} - 1$$
$$V_{\ell}(t) = \frac{1}{\ell} \left( (2\ell - 1)tV_{\ell-1}(t) - (\ell - 1)V_{\ell-2}(t) - \sqrt{2 - 2t} \right) \quad \ell = 2, 3, \ldots$$

(120) (121) (122)

Auxiliary quantities (compare Paul, 1973, Eq. 26)

$$U_{\ell}^{(1)}(t) = U_{\ell}(t) - t$$
$$U_{\ell}^{(2)}(t) = U_{\ell}(t) - 1 - t^2/2$$
$$U_{\ell}^{(3)}(t) = U_{\ell}(t) - (3\ell^2 - 1)/2$$
$$V_{\ell}^{(1)}(t) = V_{\ell}(t) - 1/2 - t/3$$
$$V_{\ell}^{(2)}(t) = V_{\ell}(t) - 1/3$$
$$U_{\ell}^{(1)}(t) = U_{\ell+1}(t)$$
$$U_{\ell}^{(2)}(t) = U_{\ell+1}(t) + 1/(\ell - 1)$$
$$U_{\ell}^{(3)}(t) = U_{\ell+1}(t) + 1/\ell + t/(\ell + 1)$$
$$U_{\ell+1}^{(1)}(t) = U_{\ell+2}(t) + 1/(\ell + 1) + t/\ell + (3\ell^2 - 1)/2(\ell - 1)$$
$$V_{\ell}^{(1)}(t) = V_{\ell+1}(t) - 1/\ell - t/(\ell + 1) - (3\ell^2 - 1)/2(\ell + 2)$$
$$V_{\ell}^{(2)}(t) = V_{\ell}(t) - 1/(\ell + 1) - t/(\ell + 2)$$
$$W(t) = 3/2 - t^2/2 - \sqrt{2 - 2t}$$

(123) (124) (125) (126) (127) (128) (129) (130) (131) (132) (133) (134)

Here the constant is either $c_f = \gamma$ for gravity anomaly or $c_f = R$ for geoid undulation and $\lambda_f^{(1)}$, $\lambda_f^{(2)}$ are defined by Eq. (41). The $Q_{\ell}^{(0)}$ truncation error coefficients characterize the truncation error for
Table 3: Truncation error coefficient computation formulas for the Eötvös kernels (isotropic Green functions) — Part 2.
The following abbreviations are used: $s := \sin(\psi_0/2)$, $t := \cos \psi_0$, where $\psi_0$ is the truncation cap radius. The t-dependence of functions is omitted for simplicity, e.g. $U'_{t+1} = U'_{t+1}(t)$.

Eötvös kernel for gravity anomaly $\Delta y$ — gradient part $\delta \mathbf{T}^{(1)}$

$$
K_0(s) = 6s(1 + s) - 4s^2 \ln(1 + 1/s) + 6 \ln(s + 1) - 2s^2 \ln(s + s^2)
$$

$$
K_1(s) = s^2(3 + 2s - 5s^2) + 4s^2(s^2 - 1) \ln(1 + 1/s) + 2s^3(s^2 - 1) \ln(s + s^2)
$$

$$
K_t(s) = \frac{1}{(2\ell + 1)F} \left( (P_{t+1} - P_{t-1}) \left[ - (\ell - 1)U'_{t+1} - (\ell + 2)V'_t - (2\ell + 1)(U'_t - 2U'_0) \right] 
+ P_t \left[ (\ell - 1)(U'_{t+2} - U'_t) - (\ell + 2)(V'_{t+1} - V'_{t+1}) + 6W - \frac{(\ell - 1)(2\ell + 2 + \ell + 1)}{(2\ell + 1)(\ell + 1)}(P_{t+1} - P_{t-1}) \right] \right) 
+ \frac{\ell - 1}{\ell(\ell + 1)}R_{t,\ell}, \quad \ell > 2
$$

Eötvös kernel for gravity anomaly $\Delta y$ — shear part $\delta \mathbf{T}^{(2)}$

$$
K_0(s) = s(3s^2 + 2s - 5) + 5 \ln(s + 1) - (1 + 3s^2)s^2 \ln(1 + 1/s) - s^3 \ln(s + s^2)
$$

$$
K_1(s) = s(1 - s - 5s^2 - s^3 - 4s^5) - s^2(1 + 2s - 4s^2) \ln(1 + 1/s) - \ln(s + 1) + (s^2 - 1)s^2 \ln(s + s^2)
$$

$$
K_t(s) = \frac{1}{(2\ell + 1)F} \left( (P_{t+1} - P_{t-1}) \left[ - \frac{(\ell - 1)}{\ell + 2}U'_{t+1} + \frac{(\ell + 2)}{\ell - 1}V'_t - \frac{(2\ell + 1)}{2}(U'_t - 4U'_0) \right] - \frac{3\ell(\ell + 1)(2\ell + 1)}{2(\ell - 1)(\ell + 2)}V'_t 
+ P_t \left[ \frac{(\ell - 1)}{\ell + 2}(U'_{t+2} - U'_t) + \frac{(\ell + 2)}{\ell - 1}(V'_{t+1} - V'_{t+1}) + 4W 
- \frac{(2\ell + 1)}{\ell(\ell + 1)(\ell + 2)}(P_{t+1} - P_{t-1}) \right] \right) 
+ \frac{\ell - 1}{\ell(\ell + 1)(\ell + 2)}R_{t,\ell}, \quad \ell > 2
$$

Eötvös kernel for incremental potential $\Delta y$ — gradient part $\delta \mathbf{V}^{(1)}$

$$
K_0(s) = s(4 - 3s - s^3) + 4(s^2 - 1) \ln(s + 1)
$$

$$
K_1(s) = s^2(4/3s^4 + s^2 - 4/3s - 1) - 4s^2(s^2 - 1) \ln(s + 1)
$$

$$
K_t(s) = \frac{1}{(2\ell + 1)F} \left( (P_{t+1} - P_{t-1}) \left[ - U'_{t+1} + V'_t + (2\ell + 1)(U'_t - 2U'_0) \right] 
+ P_t \left[ U'_{t+2} - U'_t + V'_{t+1} - V'_{t+1} - 4W - \frac{(2\ell + 2 + \ell + 1)}{(2\ell + 1)(\ell + 1)}(P_{t+1} - P_{t-1}) \right] \right) 
+ \frac{1}{\ell(\ell + 1)}R_{t,\ell}, \quad \ell > 2
$$

Eötvös kernel for incremental potential $\Delta y$ — shear part $\delta \mathbf{V}^{(2)}$

$$
K_0(s) = s(3 - 5/3s - s^2 - 1/3s^3) - \ln(s + 1) + \ln s^2 + \ln(s + s^2) + (s^2 - s^2 - 2) \ln(1 + 1/s)
$$

$$
K_1(s) = s(-1/3 - 19/9s^2 + 2/3s^3 + 4/3s^4 + 4/9s^5) + s^2(1 - s^3) \ln(s + s^2) + 1/6 \ln s^2 + (1/3 + s^2 - 4/3s^5) \ln(1 + 1/s)
$$

$$
K_t(s) = \frac{1}{(2\ell + 1)F(\ell + 2)} \left( (P_{t+1} - P_{t-1}) \left[ - (\ell - 1)U'_{t+1} - (\ell + 2)V'_t + \frac{(2\ell + 1)}{2}[(\ell - 1)(\ell + 2)(U'_t - 2U'_0) + (\ell + 1)V'_t] \right] 
+ P_t \left[ (\ell - 1)(U'_{t+2} - U'_t) + (\ell + 2)(V'_{t+1} - V'_{t+1}) - \frac{8}{3}(\ell - 1)(\ell + 2)W 
+ \frac{(2\ell + 1)^2}{3}(U''_t - V'_t) - \frac{(\ell - 1)(2\ell + 2 + \ell + 1)}{(2\ell + 1)(\ell + 1)}(P_{t+1} - P_{t-1}) \right] \right) 
+ \frac{1}{\ell(\ell + 1)(\ell + 2)}R_{t,\ell}, \quad \ell > 2
$$
the gravity field quantity \( f \), and \( \sigma^2 \) denotes potential degree variances of the gravity field model.

When we use the potential degree variances of a realistic Earth gravity field model up to a certain maximum harmonic degree, then we are able to compute the rms. truncation error by taking the square root of \( f^2 \) in Eq. (147). For example when the GPM98CR model of Wenzel (1998) was used from the harmonic degree 2 up to the maximum harmonic degree 720, we can see the rms. truncation errors for three cases in Figure 11. These three cases are the (i) Stokes and (ii) Eötvös kernels for geoid undulation and (iii) the Eötvös kernel for gravity anomaly.

By inspecting these functions of the truncation error, we see how much it differs the case (ii) from the other two cases (i) and (iii) near the origin (zero truncation radius). The truncation error remains almost constant for case (ii) there while in cases (i) and (iii) it has a steep decrease at the origin. We emphasize, that this essential difference is governed by the boundedness property of the corresponding isotropic Green (kernel) functions. As an important conclusion one can say that the truncation characteristics of the Eötvös kernel (ii) for the potential disturbance make it problematic to use it in local geoid determination. Even when we remove a very high harmonic degree reference field say up to 2000 and truncate at \( \psi = 2^\circ \), the truncation error remains \( \pm 8 \) cm, whereas Stokes integration has a truncation error with the same conditions only about \( \pm 2 \) mm. On the other hand we see that the Eötvös kernel (iii) for gravity anomalies has a favourable truncation character for small truncation radii, and the resulting gravity anomalies can be combined with terrestrial gravity anomalies in geoid determination.

6 Computational aspects of the solution and summary

One of our main ideas when developing the Eötvös gradiometric boundary value problem was to have a tool to transfer the gravity field information of torsion balance measurements into gravity field quantities, which are more directly useable for geodesy, that is gravity anomalies and geoid undulations. But it is true that the four observables of the torsion balance form only a subset of full tensor gradiometry. Therefore in this respect the Eötvös overdetermined problem is connected directly to full tensor gradiometric measurements of any kind (airborne or satellite). Unfortunately the GOCE satellite gradiometer will provide certain elements of the Eötvös tensor with only degraded accuracy due to the fact that the calibration of its capacitive gradiometer has to be performed in the presence of the Earth’s strong gravity field (Sneeuw et al., 2002). Hence the GOCE measurements cannot be inputted directly into the Eötvös gradiometric boundary value problem. On the other hand the diagonal elements (in the local orthonormal satellite-fixed frame aligned with the satellite orbit) will be provided with high accuracy by the GOCE, and Petrovskaya and Zieliński (1998) has already studied that kind of gradiometric boundary value problems.

In any case, working with formulas like the integral expressions of Eqs. (78), (79), several computational issues arise almost immediately. The first issue is the unavoidable discretization of the integration, the second one is its restriction instead of the whole reference sphere to a limited spatial domain where data are available.

The first issue is addressed by introducing finite elements on the sphere \( S^2 \), and thus we approximate integration by summation over these finite elements. In the Eötvös gradiometric problem we face the problem of azimuth dependency. This means that the summation weights are strongly azimuth dependent for each of the four observables. Moreover, with the usual discretization over constant latitude, longitude blocks, the symmetry of these weights with respect to the NW or SE directions is lost because of the \( \cos 2\alpha \) factor in the integrand. As a consequence we have integration weights, especially for non-square shaped central blocks, which are strongly dependent on the shape of these finite elements.
This next problem can be solved by a suitable integration process over the blocks, for example by position- and shape-dependent subdivision scheme. The idea is similar to the scheme discussed by de Min (1994) in case of the Stokes integral.

The second problem, the spatial truncation requires to be solved by either a suitable remove-restore procedure, or by estimating and explicitly adding the truncated part from a geopotential model. As it was shown by Sjöberg and Hunegnaw (2000), the two approaches are equivalent. We prefer the second one, since in this case no data reduction (remove) step is necessary prior to the integration. When we know the spectral coefficients of the geopotential $c_{lm}$, up to the maximum harmonic degree $M$, the truncation error of the Eötvös integrals is estimated for the truncation radius $\phi_0$ by the following sum

$$\Delta_{\phi_0} = \frac{\gamma}{2} \sum_{l=2}^{M} \sum_{m=0}^{l} c_{lm} Y_{lm} \psi(\phi_0) \lambda_{l}^{(i)} \sum_{m=0}^{l} c_{lm} Y_{lm}, \quad i \in \{1, 2\}$$

(148)

Synthetic gravity field data has already been proved useful for several tasks in gravity field determination. We mention here for example software validation (Featherstone and Olliver, 1997), plumbline computation (Papp and Benedek, 2000), methodology testing (Featherstone, 1999). Both geopotential based functional representation models (Haagmans, 2000, see for example) and source (forward) synthetic models (Papp and Kalmár, 1996) provide analytical relations for gravity field parameters. Therefore either of the above models can generate gravity tensor components, gravity anomalies and geoid heights.

By addressing both important issues mentioned above, we have developed a software to compute gravity anomalies from torsion balance measurements and our goal is to validate this software which is in a testing phase now.

As a first check, we generated at each point of a 181 rows by 169 columns grid 122 356 gravity gradients from the EGM96 geopotential coefficients as well as 30 589 gravity anomalies over the region $40^o < \phi < 55^o$ of geodetic latitude $\phi$ and $12^o < \lambda < 33^o$ geodetic longitude $\lambda$. The program geopot97 by Smith (1998) was used for this purpose since the full gravity gradient tensor can be computed by it. Gravity gradients can be transformed by the Eötvös integrals Eqs. (78), (79) to gravity anomalies by taking into account the truncation error estimate in Eq. (148). The above geopotential-based analytical model may give checks for the resulting gravity anomalies. As an illustration we show the truncation error estimate Eq. (148) for this area with truncation cap radius $\phi_0 = 3.8^o$, in Figure 12. After solving various issues of the numerical integration successfully we will present computational results in a forthcoming paper.

Another choice to validate our software is the forward litospheric density model developed by Papp
Fig. 12: Estimate of the truncation error of gravity anomalies for the Eötvös integrals (see Eq. 148). The truncation cap radius is $\varphi_0 = 3.8^\circ$. The EGM96 geopotential model is used up to the harmonic degree 360. Units are mGals.

and Kalmár (1996)). This model is promising since the generated gravity field is not only analytically correct but it is sufficiently close to the real gravity field in the Pannonian Basin. The litospheric model can also produce the full gravity gradient tensor at any point and gravity anomalies as well. There is a promising cooperation between various institutions in Hungary to generate and use such synthetic data. Therefore we hope that we can present results in the near future by using the synthetic gravity field data of this litospheric model as well.

Finally now we summarize the main results of this paper. We have shown first that two specific combinations of Eötvös torsion balance measurements, one vectorial and a second one tensorial, lead to a simple solution of this overdetermined gradiometric boundary value problem in the spectral domain of three different sets of spherical harmonic basis functions. This is true either for gravity anomalies or geoid undulations as the target.

Second, we have shown that the solution can be expressed in the spatial domain by the Eötvös integrals, one for each incremental observable combination. These integrals are similar to the Stokes integral in two respects. First, each of them contains an isotropic kernel function (Eötvös kernel) that can be expressed in closed form. Second, the Eötvös kernels have the same kind of singularity at the origin as the Stokes function. But, as we have shown, this statement is not true when the target is directly the incremental potential (or geoid undulation), since the corresponding kernels are not singular. The difference between the Stokes and Eötvös integrals is that the latter are also azimuth dependent at the source point. We mention that this azimuth dependency is different from the azimuth dependency of the Vening-Meinesz integral, where the azimuth at the evaluation point enters into the expressions.

Third, we have explicitly shown by deriving a truncation coefficient computation scheme of the Eötvös kernels that it is impracticable to transform spatially restricted gradient data of the torsion balance into geoid undulations directly; instead, we propose a two-step integration approach to be followed via gravity anomalies. This conclusion holds true also for space gradiometry using the same gravity gradient data.

Now we finish our paper by several remarks. The discussion about the Eötvös gradiometric boundary value problem is extensible in many respects. It seems relatively easy to include the fifth independent observable of the Eötvös tensor, the second vertical derivative of the geopotential into our approach. This will lead in spherical approximation to Green functions that are expressible by elliptic integrals. Moreover, it would be advantageous to leave the spherical approximation behind and define everything rigorously on the International Reference Ellipsoid (IRE). The downward continuation operators should also be constructed to solve the downward continuation problem of gravity gradient observables to the reference sphere or ellipsoid. Finally we mention here that the study of truncation properties of the future GOCE observables would also be desirable.

All these problems open us new fields of research and it is nice to see how the original ideas of Loránd Eötvös, conceived more than hundred years ago, are interconnected with modern research and hopefully they will lead us to the better knowledge of the Earth’s gravity field.

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Appendix A Addition theorems of spheroidal vector and shear tensor harmonics

In this Appendix we prove the addition theorems of spheroidal vector (Eq. 32) and shear tensor (Eq. 37) spherical harmonics. These addition theorems Eqs.
(62), (63) are needed for the multiplicative decompositions of the vector- respectively tensor-valued Green functions (Eötvös functions) Eqs. (66), (67).

The following steps lead us to the proofs. First, we define the spin-$s$ spherical harmonics and elucidate their connection with vector- and tensor-valued spherical harmonics. Second, we will reveal the addition theorem of spin-$s$ harmonics by virtue of their relation to the $D$-functions. Finally the connections between the spin-1 and spin-2 spherical harmonics with vector and tensor harmonics lead us to the desired addition theorems.

We start by defining the spin-$s$ spherical harmonics $sY^m_l(\phi, \lambda)$ by their relation to the $D$-functions (Goldberg et al., 1967) as

$$sY^m_l(\phi, \lambda) := \sqrt{\frac{2\ell + 1}{4\pi}} \mathcal{D}^m_{l,m}(\lambda, \phi, 0) \quad (A1)$$

where the $\mathcal{D}^m_{l,m}(\alpha, \beta, \gamma)$ functions, ordered as matrices form group representations for the three-dimensional rotation group SO(3) (Risbo, 1996). Explicitly writing out the $D$-functions Eq. (A1) becomes (Hu and White, 1997)

$$sY^m_l(\phi, \lambda) = \sqrt{\frac{(\ell + s)!}{(\ell - s)!}} (-1)^{s-r} e^{i m \phi} \cos \frac{\phi}{2} \sum_r \left(\begin{array}{c} \ell - s \\ r \end{array}\right) \left(\begin{array}{c} \ell + s \\ r + s - m \end{array}\right) (-1)^{\ell - r - s} e^{i m \phi} \cos \frac{\phi}{2} r^{2s+m} \quad (A2)$$

By virtue of their relation to rotation matrices, the spin-$s$ harmonics satisfy the compatibility relation with spherical harmonics $sY^m_l = Y^m_l$ and form a complete orthonormal system for spin-$s$ quantities $s\mathbf{f}$ on the unit sphere $S^2$. We note that a complex function $s\mathbf{f}(\phi, \lambda)$ defined on $S^2$ is said to have spin $s$ if under a right-handed rotation of the $\{e_1, e_2 \mid P\}$ frame at $P \sim (\phi, \lambda)$ by an angle $\psi$ it transforms as $s\mathbf{f}(\phi, \lambda) = e^{-i s \psi} s\mathbf{f}(\phi, \lambda)$.

To establish the connection of spin-$s$ harmonics and the spheroidal vector and shear tensor harmonics we introduce the complex orthonormal basis $\{\mathbf{m}_s, \mathbf{m}_s \mid P\}$ in the local tangent space $T_P S^2$ as

$$\mathbf{m}_s := \frac{1}{\sqrt{2}} (\mathbf{e}_1 \mp i \mathbf{e}_2). \quad (A3)$$

The $\{\mathbf{e}_1, \mathbf{e}_2 \mid P\}$ frame is shown in Figure 1. Now if we act with the surface gradient operator $\nabla_{surf}$ (Eq. 9) on the spherical harmonic $Y^m_l$, we get the following expression in terms of the spin-1 spherical harmonics (compare Hu, 2000, Eq. 56)

$$\nabla_{surf} Y^m_l = -\sqrt{\frac{(\ell + 1)}{2}} [Y^m_l(\mathbf{m}_s - \mathbf{m}_s) \mathbf{m}_s - Y^m_l(\mathbf{m}_s) \mathbf{m}_s]. \quad (A4)$$

Note that our definition of $\mathbf{m}_s$ differs from that of Hu (2000) because we use a $(\phi, \lambda)$-system insted of the $(\theta, \lambda)$-system, where $\theta$ denotes the spherical polar distance on $S^2$. If now we compare the above formula with Eq. (32) definition of spheroidal vector harmonics, we finally get the following connection

$$S^m_l(\phi, \lambda) = -\frac{1}{\sqrt{2}} [Y^m_l(\phi, \lambda) \mathbf{m}_s - Y^m_l(\phi, \lambda) \mathbf{m}_s]. \quad (A5)$$

between spheroidal vector and spin-1 spherical harmonics.

Next we derive a similar relation of shear tensor harmonics $Z^m_l$ and spin-2 spherical harmonics $s^2Y^m_l$. Again we define the following complex basis tensors

$$\mathbf{M}_s := \mathbf{m}_s \otimes \mathbf{m}_s \quad (A6)$$

in the local tangent space $T_P S^2 \otimes T_P S^2$ of $S^2$. In the $\{e_1 \otimes e_j, \ i, j \in \{1, 2\} \mid P\}$ basis these tensors have the following components in an obvious matrix notation

$$\mathbf{M}_s := \frac{1}{2} \left( \begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array} \right) = \frac{1}{2} (e_1 \otimes e_1 - e_2 \otimes e_2) \quad \mp i (e_1 \otimes e_2 + i e_2 \otimes e_1). \quad (A7)$$

The spin-2 harmonics are related to the ordinary complex spherical harmonics (compare Hu and White, 1997, Eq. 3) as

$$s^2Y^m_l(\phi, \lambda) = \sqrt{\frac{\ell(\ell + 1)}{(\ell + 2)(\ell + 3)}} \left( \nabla^2_{\phi\phi} - \nabla^2_{\lambda\lambda} \mp 2 i \nabla^2_{\phi\lambda} \right) Y^m_l(\phi, \lambda) \quad (A8)$$

where the $\nabla^2_{\phi\phi}$ and $\nabla^2_{\lambda\lambda}$, and $\nabla^2$ are covariant differential operators are those defined by Eqs. (15). If we take into account the relations Eq. (A8), (14) and (A7), the action of the $\nabla^2$ operator on spherical harmonics $Y^m_l$ can be expressed as

$$\nabla^2 Y^m_l(\phi, \lambda) = \sqrt{\frac{\ell(\ell + 1)!}{2(\ell - 2)!}} \left[ s^2Y^m_l(\phi, \lambda) \mathbf{M}_s + s_2Y^m_l(\phi, \lambda) \mathbf{M}_s \right]. \quad (A9)$$

Now by comparing the above relation with the definition of shear tensor harmonics $Z^m_l$ in Eq. (37) we again get the desired connection with spin-2 spherical harmonics as

$$Z^m_l(\phi, \lambda) = \frac{1}{\sqrt{2}} \left[ s^2Y^m_l(\phi, \lambda) \mathbf{M}_s + s_2Y^m_l(\phi, \lambda) \mathbf{M}_s \right]. \quad (A10)$$
In the second step of this Appendix we introduce the addition theorem of spin-s spherical harmonics. We follow the discussion of Hu and White (1997). This addition theorem follows from the relation Eq. (A1) between spin-s harmonics and rotations. From the geometry of Figure 13 it can be seen that a rotation from \( P^s(\phi^*, \lambda^*) \) through the pole (origin) \( N \) to \( P(\phi, \lambda) \) is equivalent to a direct rotation by the Euler angles \( (\alpha, \beta, \gamma) \). Thus the generalized addition relation of spin-s harmonics follows from the group multiplication property of rotation matrices as

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} P^m_{\ell}(\phi^*, \lambda^*) Y^m_{\ell}(\phi, \lambda) \\
= \sqrt{\frac{2\ell + 1}{4\pi}} Y^0_{\ell} (\hat{z} - \beta, \alpha) e^{-i\gamma}. 
\end{align*}
\]  

(A11)

By setting \( s_2 = 0 \) in this equation and introducing the notations of Figure 6 yields

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} P^m_{\ell}(\phi^*, \lambda^*) Y^m_{\ell}(\phi, \lambda) \\
= \sqrt{\frac{2\ell + 1}{4\pi}} Y_{\ell,-s} (\hat{z} - \psi, 2\pi - \alpha^*).
\end{align*}
\]  

(A12)

Now if we take into account the definition of \( Y_{\ell s} \) Eq. (23) and its symmetry Eq. (24) this equation becomes the following relation

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} P^m_{\ell}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \\
= \sqrt{\frac{2\ell + 1}{4\pi}} (-1)^s \sqrt{\frac{(\ell - s)!}{(\ell + s)!}} Y_s^{\ell m}(\cos \psi) \ e^{is\alpha^*},
\end{align*}
\]

(A13)

since \( e^{i(2\pi - \alpha')} = e^{-i(2\pi - \alpha')} = e^{is\alpha^*} \).

The last easy step of deriving the desired addition theorems is to note the following decompositions through Eqs. (A5), (A10)

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} S^m_{\ell m}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \\
= \sum_{m=-\ell}^{\ell} P^m_{\ell}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \frac{m_z}{\sqrt{2}} \quad (A14)
\end{align*}
\]

and

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} Z^m_{\ell m}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \\
= \sum_{m=-\ell}^{\ell} 2 P^m_{\ell}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \frac{M_z}{\sqrt{2}} \quad (A15)
\end{align*}
\]

since \( \overline{m}_z = m_z \) and \( \overline{M}_z = M_z \). The four sums on the right side of these equations can be computed from the general addition theorem Eq. (A13) by setting \( s = \pm 1 \) and \( s = \pm 2 \). This leads to the following addition theorems

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} S^m_{\ell m}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \\
= \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!}} P^m_{\ell}(\cos \psi) \frac{1}{\sqrt{2}} [e^{i\alpha^*} m_z - e^{-i\alpha^*} m_-] \quad (A16)
\end{align*}
\]

and

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} Z^m_{\ell m}(\phi^*, \lambda^*) Y_{\ell m}(\phi, \lambda) \\
= \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} P^m_{\ell}(\cos \psi) \frac{1}{\sqrt{2}} [e^{i\alpha^*} M_- + e^{-i\alpha^*} M_+] \quad (A17)
\end{align*}
\]
The terms in square brackets can be evaluated if we take into account the relations Eq. (A3) and (A6). This gives us finally the following equations

\[ \sum_{n=-\ell}^{\ell} s_{\ell n}(\phi', \lambda')Y_{\ell m}(\phi, \lambda) = \]

\[ = \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!}} P^1_\ell(\cos \psi)(\cos \alpha^* e_1 + \sin \alpha^* e_2) \]

(A18)

\[ \sum_{n=-\ell}^{\ell} z_{\ell n}(\phi', \lambda')Y_{\ell m}(\phi, \lambda) = \]

\[ = \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 1)!}{(\ell + 2)!}} P^2_\ell(\cos \psi) - \frac{1}{\sqrt{2}}(\cos 2\alpha^* e_1 \otimes e_1 - \cos 2\alpha^* e_2 \otimes e_2 - \sin 2\alpha^* e_1 \otimes e_2 - \sin 2\alpha^* e_2 \otimes e_1). \]

(A19)

Since these are the same as the required addition relations Eqs. (62) and (63), our proof is now complete.

**Appendix B Closed-form expressions of isotropic Green functions (Eötvös kernels)**

In this Appendix we derive closed-form expressions for the isotropic part of Green functions for gradient Eq. (72) and shear (73) observable combinations of the Eötvös overdetermined gradimetric boundary value problem. The following three steps will lead us to reach the above goal. First, we (i) manage the partial fractional decompositions of the general term of infinite series. Next we (ii) substitute closed-form expressions of these infinite series with ordinary Legendre polynomials and sum up the partial fractions. Finally we will (iii) take derivatives of order 1 and 2 of the closed-form expressions and this will give us the desired analytical expression of infinite series with Legendre functions of order 1 and 2.

**Step (i)** Our goal is to derive closed-form expressions of the following Eötvös kernels, Eqs. (72) and (73)

\[ \frac{1}{\ell} \quad \frac{1}{\ell + 1} \quad \text{and} \quad \frac{1}{\ell + 2} \]

(partial fractional decomposition), and here they are:

\[ \frac{\ell - 1}{\ell(\ell + 1)} = \frac{2}{\ell} - \frac{1}{\ell} \]

(B3)

\[ \frac{\ell - 1}{\ell(\ell + 1)(\ell + 2)} = -\frac{1}{\ell} + \frac{2}{\ell + 1} - \frac{3}{2} \frac{1}{\ell + 2}. \]

(B4)

**Step (ii)** We have the following known sums (compare Moritz, 1980, Chap 23) of Legendre polynomials

\[ S_1(s, \psi) := \sum_{\ell=2}^{\infty} s^{\ell+2} \frac{1}{\ell} P_\ell(\cos \psi) \]

\[ = s^2 \ln \frac{2}{N} - s^3 \cos \psi \]

(B5)

\[ S_2(s, \psi) := \sum_{\ell=2}^{\infty} s^{\ell+1} \frac{1}{\ell + 1} P_\ell(\cos \psi) \]

\[ = s \ln \left( \frac{2s}{1 - s + L} \right) - s^2 - \frac{1}{2} s^3 \cos \psi \]

(B6)

\[ S_3(s, \psi) := \sum_{\ell=2}^{\infty} s^{\ell+2} \frac{1}{\ell + 2} P_\ell(\cos \psi) \]

\[ = L - 1 + \cos \psi \ln \left( \frac{2s}{1 - s + L} \right) - \frac{1}{2} s^2 - \frac{1}{3} s^3 \cos \psi \]

(B7)

where we introduced the abbreviations

\[ s := \frac{r^*}{r} \]

\[ L := \sqrt{1 - 2s \cos \psi + s^2} \]

\[ N := 1 - s \cos \psi + L. \]

We introduce the functions \( K^{(1)}(s, \psi) \) and \( K^{(2)}(s, \psi) \) that are series of ordinary Legendre polynomials defined according to

\[ K^{(1)}(s, \psi) = \sum_{\ell=2}^{\infty} s^{\ell+2} \frac{\ell - 1}{\ell(\ell + 1)} P_\ell(\cos \psi) \]

(B8)

\[ K^{(2)}(s, \psi) = \sum_{\ell=2}^{\infty} s^{\ell+2} \frac{\ell - 1}{\ell(\ell + 1)(\ell + 2)} P_\ell(\cos \psi). \]

(B9)
These series can be summed up easily by taking into account Eqs. (B3) and (B4)

\[ K^{(1)}(s, \psi) = 2S_2(s, \psi) - S_1(s, \psi) \quad (B10) \]

\[ K^{(2)}(s, \psi) = \frac{1}{2} S_3(s, \psi) + 2S_2(s, \psi) - \frac{1}{2} S_1(s, \psi). \quad (B11) \]

By substituting Eqs. (B5), (B6), (B7) we have the following results

\[ K^{(1)}(s, \psi) := 2s \ln \left( \frac{2s}{1-s+L} \right) - 2s^2 - s^2 \ln \frac{2}{N} \quad (B12) \]

\[ K^{(2)}(s, \psi) := -\frac{1}{2} s^2 \ln \frac{2}{N} + (2s - \frac{3}{2} \cos \psi) \ln \left( \frac{2s}{1-s+L} \right) - \frac{3}{4} s^2 - \frac{3}{2} L + \frac{3}{2}. \quad (B13) \]

**Step (iii)** In this final step we will derive the closed-form expressions of the Eötvös kernels \( G^{(1)}_{\Delta \psi}(s, \psi) \) and \( G^{(2)}_{\Delta \psi}(s, \psi) \) expressed as derivatives of the functions \( K^{(1)}(s, \psi) \) and \( K^{(2)}(s, \psi) \) according to their relations Eqs. (70) and (71):

\[ G^{(1)}_{\Delta \psi}(s, \psi) = -\frac{d}{ds} K^{(1)}(s, \psi) \quad (B14) \]

\[ G^{(2)}_{\Delta \psi}(s, \psi) = \left( \frac{d}{ds} \cot(\psi) \frac{d}{d\psi} \right) K^{(2)}(s, \psi). \quad (B15) \]

We have to compute the derivatives of all the terms in Eqs. (B12) and (B13). Some of them are the following ones:

\[ \frac{dL}{d\psi} = s \sin \psi / L \]

\[ \left( \frac{d}{d\psi} - \cot(\psi) \frac{d}{d\psi} \right) L = -s \sin^2 \psi / L^3 \]

\[ \frac{dN}{d\psi} = s \sin \psi / L + \frac{s}{2} \sin \psi / LN \]

\[ \frac{d}{d\psi} \ln \frac{2}{N} = -\frac{s \sin \psi}{LN} (L+1) \]

\[ \left( \frac{d}{d\psi} - \cot(\psi) \frac{d}{d\psi} \right) \ln \frac{2}{N} = \frac{s^2 \sin^2 \psi}{L^3 N^2} (L^3 + 2L^2 + L + N) \]

and after tedious but straightforward computations the following closed-form expressions

\[ G^{(1)}_{\Delta \psi}(s, \psi) = \frac{s^3 \sin \psi}{LN} (1 - L), \quad (B16) \]

\[ G^{(2)}_{\Delta \psi}(s, \psi) = \frac{s^2 \sin^2 \psi}{2L^2 N^2} \left[ s^2 (3N - L^3 + 2L^2 + 3L) - 3s \cos \psi (N + L^2 + L) - 6L^2 N + 3N^2 \right] \quad (B17) \]

will be yielded as the desired result. This completes the derivation of closed-form expressions of the Eötvös kernel functions.

**References**


