Determination of gravity anomalies from torsion balance measurements

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Abstract. There is a dense network of torsion balance stations in Hungary, covering an area of about 40000 km^2. These measurements are a very useful source to study the short wavelength features of the local gravity field, especially below 30 km wavelength. Our aim is thus to use these existing torsion balance data in combination with gravity anomalies. Therefore a method was developed, based on integration of horizontal gravity gradients over finite elements, to predict gravity anomaly differences at all points of the torsion balance network. Test computations were performed in a Hungarian area extending over about 800 km^2. There were 248 torsion balance stations and 30 points among them where Δg values were known from measurements in this test area.

Keywords. Gravity anomalies, torsion balance measurements.

1 The proposed method

Let's start from the fundamental equation of physical geodesy:

$$\Delta g = g - \gamma = -\frac{\partial T}{\partial r} - \frac{2T}{R},$$

where T is the potential disturbance and R is the mean radius of the Earth (Heiskanen, Moritz 1967). Changing of gravity anomaly Δg between two arbitrary points P_1 and P_2 is:

$$(\Delta g_2 - \Delta g_1) = -\left[\left(\frac{\partial T}{\partial r}\right)_2 - \left(\frac{\partial T}{\partial r}\right)_1\right] - \frac{2}{R}(T_2 - T_1) .$$

In a special coordinate system (x points to North, y to East and z to Down) the changing of gravity anomaly:

$$(\Delta g_2 - \Delta g_1) = \left[\left(\frac{\partial T}{\partial z}\right)_2 - \left(\frac{\partial T}{\partial z}\right)_1\right] - \frac{2}{R}(T_2 - T_1) .$$

Let's estimate the order of magnitude of term $(2/R)(T_2 - T_1)$ which is:

$$\frac{2}{R}(T_2 - T_1) = \frac{2\gamma}{R}\Delta N_{12}, \tag{1}$$

where ΔN_{12} is the changing of geoid undulation between the two points. If the changing of geoid undulation between two points is 1 m, than the value of (1) is 0.3 mGal (1 mGal = 10^{-5}m/s^2). Taking into account the average distance between the torsion balance stations and supposing not more than dm order of geoid undulation's changing, the value of (1) can be negligible.

Applying the notation $T_z = \partial T / \partial z$ for the partial derivatives, the changing of gravity anomalies between the two points P_1 and P_2 is:

$$(\Delta g_2 - \Delta g_1) = (T_z)_2 - (T_z)_1 .$$

So in the case of displacement vector $d\mathbf{r}$ the elementary change of gravity anomaly Δg will be:

$$d\Delta g = \nabla(\Delta g) \cdot d\mathbf{r} = \frac{\partial \Delta g}{\partial x}dx + \frac{\partial \Delta g}{\partial y}dy + \frac{\partial \Delta g}{\partial z}dz = T_{zx}dx + T_{zy}dy + T_{zz}dz .$$

Integrating this equation between points P_1 and P_2 we get the changing of gravity anomaly:

$$(\Delta g_2 - \Delta g_1) = \int_1^2 d\Delta g = \int_1^2 T_{zx}dx + \int_1^2 T_{zy}dy + \int_1^2 T_{zz}dz, \tag{2}$$

where $T_{zx} = W_{zx} - U_{zx}$, $T_{zy} = W_{zy} - U_{zy}$ and $T_{zz} = W_{zz} - U_{zz}$; W_{zx} and W_{zy} are horizontal gradients of gravity measured by torsion balance, W_{zz} is the measured vertical gradient, U_{zx} and U_{zy} are the normal value of horizontal gravity gradients, and U_{zz} is the normal value of vertical gradient. According to Torge (1989):

$$U_{zx} = \frac{\partial \gamma}{\partial x} = \frac{\gamma_e \beta}{M} \sin 2\varphi \ , \ U_{zy} = 0 \ ,$$
$$U_{zz} = \gamma \left(\frac{1}{M} + \frac{1}{N} \right) + 2\omega^2$$

where M and N is the curvature radius in the meridian and in the prime vertical, $\gamma = \gamma_e (1 + \beta \sin^2 \varphi)$ is the normal gravity on the ellipsoid. With the values of the Geodetic Reference System 1980, the following holds at the surface of the ellipsoid:

$$U_{zx} = 8.1 \sin 2\varphi \ ns^{-2}$$
$$U_{zz} = 3086 \ ns^{-2} \ .$$

Let's compute the first integral on the right side of equation (2) between the points P_1 and P_2. Before the integration a relocation to a new coordinate system is necessary; the connection between the coordinate systems (x,y) and the new one (u,v) can be seen on Figure 1. Denote the direction between the points P_1 and P_2 with u and be the coordinate axis v perpendicular to u. Denote the azimuth of u with α_{12} and point the z axis to down, perpendicularly to the plane of (xy) and (uv)!

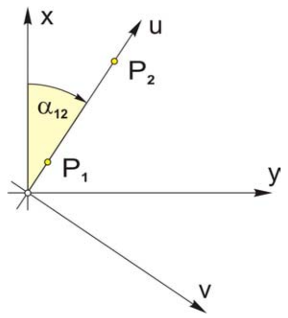

Fig. 1 Coordinate transformation (x,y)→(u,v)

The transformation between the two systems is:

$$\left. \begin{aligned} x &= u \cos\alpha_{12} - v \sin\alpha_{12} \\ y &= u \sin\alpha_{12} + v \cos\alpha_{12} \end{aligned} \right\} \ .$$

Using these equations, the first derivatives of any function W are:

$$\frac{\partial W}{\partial u} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial W}{\partial x}\cos\alpha_{12} + \frac{\partial W}{\partial y}\sin\alpha_{12}$$

$$\frac{\partial W}{\partial v} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial v} = -\frac{\partial W}{\partial x}\sin\alpha_{12} + \frac{\partial W}{\partial y}\cos\alpha_{12}$$

From this first equation, if $W = T_z$ than

$$T_{zu} du = \left(T_{zx} \cos\alpha_{12} + T_{zy} \sin\alpha_{12} \right) du = T_{zx} dx + T_{zy} dy \ ,$$

because

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\alpha_{12} \\ \sin\alpha_{12} \end{pmatrix} du \ .$$

If points P_1 and P_2 are close to each other as required, integrals on the right side of equation (2) can be computed by the following trapezoid integral approximation formula:

$$\int_1^2 \left(T_{zx} dx + T_{zy} dy \right) = \int_1^2 T_{zu} du \approx \frac{s_{12}}{2} \left[\left(T_{zu} \right)_1 + \left(T_{zu} \right)_2 \right] \quad (3)$$

$$\int_1^2 T_{zz} dz \approx \frac{\Delta h_{12}}{2} \left[\left(T_{zz} \right)_1 + \left(T_{zz} \right)_2 \right] \quad (4)$$

where s_{12} is the horizontal distance between points P_1 and P_2, and Δh_{12} is the height difference between these two points.

The value of integral (4) depends on the vertical gradient disturbance T_{zz} and the height difference between the points. If points are at the same height (on a flat area) and in case of small vertical gradient disturbances the third integral in (2) can be neglected. (E.g. the value of (4) is 0.25 mGal in case of $\Delta h_{12} = 50m$ and $[(T_{zz})_1 + (T_{zz})_2]/2 = 50E$).

So, discarding the effect of (4) the differences of gravity anomalies between two points can be computed by the approximate equation:

$$(\Delta g_2 - \Delta g_1) \approx \frac{s_{12}}{2} \left\{ \begin{aligned} & [(T_{zx})_1 + (T_{zx})_2] \cos\alpha_{12} \\ & + [(T_{zy})_1 + (T_{zy})_2] \sin\alpha_{12} \end{aligned} \right\} \ . \quad (5)$$

2 Practical solutions

If we have a large number of torsion balance measurements, it is possible to form an interpolation net (a simple example can be seen in Figure 2) for determining gravity anomalies at each torsion balance points (Völgyesi, 1993, 1995, 2001). On the basis of Eq. (5)

$$(\Delta g_k - \Delta g_i) = C_{ik} \quad (6)$$

can be written between any adjacent points, where

$$C_{ik} = s_{ik}\left\{ \frac{(W_{zx} - U_{zx})_i + (W_{zx} - U_{zx})_k}{2} \cos\alpha_{ik} + \frac{(W_{zy})_i + (W_{zy})_k}{2} \sin\alpha_{ik} \right\} . \tag{7}$$

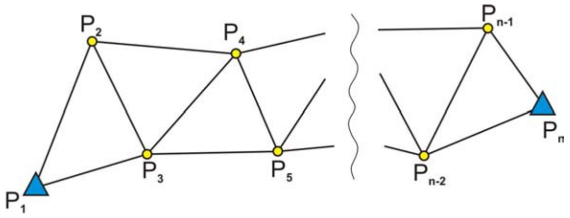

Fig. 2 Interpolation net connecting torsion balance points

For an unambiguous interpolation it is necessary to know the real gravity anomaly at a few points of the network (triangles in Figure 2). Let us see now, how to solve interpolation for an arbitrary network with more points than needed for an unambiguous solution, where gravity anomalies are known. In this case the Δg values can be determined by adjustment.

The question arises what data are to be considered as measurement results for adjustment: the real torsion balance measurements W_{zx} and W_{zy} , or C_{ik} values from Eq. (7). Since no simple functional relationship (observation equation) with a measurement result on one side and unknowns on the other side of an equation can be written, computation ought to be made under conditions of adjustment of direct measurements, rather than with measured unknowns – this is, however, excessively demanding in terms of storage capacity. Hence concerning measurements, two approximations will be applied: on the one hand, gravity anomalies from measurements at the fixed points are left uncorrected – thus, they are input to adjustment as constraints – on the other hand, C_{ij} on the left hand side of fundamental equation (6) are considered as fictitious measurements and corrected. Thereby observation equation (6) becomes:

$$C_{ik} + v_{ik} = \Delta g_k - \Delta g_i \tag{8}$$

permitting computation under conditions given by adjusting indirect measurements between unknowns (Detrekői, 1991).

The first approximation is possible since reliability of the gravity anomalies determined from measurements exceeds that of the interpolated values considerably. Validity of the second approximation will be reconsidered in connection with the problem of weighting.

For every triangle side of the interpolation net, observation equation (8):

$$v_{ik} = \Delta g_k - \Delta g_i - C_{ik} \tag{9}$$

may be written. In matrix form:

$$\underset{(m,1)}{\mathbf{v}} = \underset{(m,2n)}{\mathbf{A}}\,\underset{(2n,1)}{\mathbf{x}} + \underset{(m,1)}{\mathbf{l}}$$

where **A** is the coefficient matrix of observation equations, **x** is the vector containing unknowns Δg , **l** is the vector of constant terms, *m* is the number of triangle sides in the interpolation net and *n* is the number of points. The non-zero terms in an arbitrary row *i* of matrix **A** are:

$$[\ldots \quad 0 \quad +1 \quad -1 \quad 0 \quad \ldots]$$

while vector elements of constant term **l** are the C_{ik} values.

Gravity anomalies fixed at given points modify the structure of observation equations. If, for instance, $\Delta g_k = \Delta g_{k0}$ is given in (8), then the corresponding row of matrix **A** is:

$$[\ldots \quad 0 \quad 0 \quad -1 \quad 0 \quad \ldots]$$

the changed constant term being: $C_{ij} - \Delta g_{k0}$, that is Δg_k , and of coefficients of Δg_k are missing from vector **x**, and matrix **A**, respectively, while corresponding terms of constant term vector **l** are changed by a value Δg_{k0} .

Adjustment raises also the problem of weighting. Fictive measurements may only be applied, however, if certain conditions are met. The most important condition is the deducibility of covariance matrix of fictive measurements from the law of error propagation, requiring, however, a relation yielding fictive measurement results, – in the actual case, Eq. (7). Among quantities on the right-hand side of (7), torsion balance measurements W_{zx} and W_{zy} may be considered as wrong. They are about equally reliable $\pm 1E$ ($1E = 1\textit{Eötvös Unit} = 10^{-9}\,s^{-2}$), furthermore, they may be considered as mutually independent quantities, thus, their weighting coefficient matrix $\mathbf{Q}_{WW}$ will be a unit matrix. With the knowledge of $\mathbf{Q}_{WW}$, the weighting coefficient matrix $\mathbf{Q}_{CC}$ of fictive measurements C_{ik} after Detrekői (1991) is:

$$\mathbf{Q}_{CC} = \mathbf{F}^*\mathbf{Q}_{WW}\mathbf{F} = \mathbf{F}^*\mathbf{F}$$

$\mathbf{Q}_{WW} = \mathbf{E}$ being a unit matrix. Elements of an arbitrary row i of matrix $\mathbf{F}^*$ are:

$$\left[\left(\frac{\partial C_{ik}}{\partial W_{zx}}\right)_1 \left(\frac{\partial C_{ik}}{\partial W_{zx}}\right)_2 ,\ldots, \left(\frac{\partial C_{ik}}{\partial W_{zx}}\right)_n \right.$$
$$\left. \left(\frac{\partial C_{ik}}{\partial W_{zy}}\right)_1 \left(\frac{\partial C_{ik}}{\partial W_{zy}}\right)_2 ,\ldots, \left(\frac{\partial C_{ik}}{\partial W_{zy}}\right)_n \right]$$

For the following considerations let us produce rows $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ of matrix $\mathbf{F}^*$ (referring to sides between points $P_1 - P_2$ and $P_1 - P_3$ respectively):

$$\mathbf{f}_1^* = \left[\frac{s_{12}\sin\alpha_{12}}{2}, \frac{s_{12}\sin\alpha_{12}}{2}, 0, 0, \ldots, 0, \right.$$
$$\left. \frac{s_{12}\cos\alpha_{12}}{2}, \frac{s_{12}\cos\alpha_{12}}{2}, 0, 0, \ldots, 0 \right]$$

and

$$\mathbf{f}_1^* = \left[\frac{s_{13}\sin\alpha_{13}}{2},\ 0, \frac{s_{13}\sin\alpha_{13}}{2}, 0, 0, \ldots, 0, \right.$$
$$\left. \frac{s_{13}\cos\alpha_{13}}{2},\ 0, \frac{s_{13}\cos\alpha_{13}}{2}, 0, 0, \ldots, 0 \right]$$

Using $\mathbf{f}_1^*$, variance of C_{ik} value referring to side $P_1 - P_2$ is:

$$m^2 = \frac{s_{12}^2}{4}\left(2\sin^2\alpha_{12} + 2\cos^2\alpha_{12}\right) = \frac{s_{12}^2}{2}$$

while $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ yield covariance of C_{ik} values for sides $P_1 - P_2$ and $P_1 - P_3$:

$$\mathrm{cov} = \frac{s_{12}s_{13}}{4}\left(\sin\alpha_{12}\sin\alpha_{13} + \cos\alpha_{12}\cos\alpha_{13}\right).$$

Thus, fictive measurements may be stated to be correlated, and the weighting coefficient matrix contains covariance elements at the junction point of the two sides. If needed, the weighting matrix may be produced by inverting this weighting coefficient matrix. Practically, however, two approximations are possible: either fictive measurements C_{ij} are considered to be mutually independent, so weighting matrix is a diagonal matrix; or fictive measurements are weighted in inverted quadratic relation to the distance.

By assuming independent measurements, the second approximation results also from inversion, since terms in the main diagonal of the weighting coefficient matrix are proportional to the square of the side lengths. The neglection is, however, justified, in addition to the simplification of computation, also by the fact that contradictions are due less to measurement errors than to functional errors of the computational model (Völgyesi, 1993).

3 Test computations

Test computations were performed in a Hungarian area extending over about 800 km^2. In the last century approximately 60000 torsion balance measurements were made mainly on the flat territories of Hungary, at present 22408 torsion balance measurements are available. Location of these 22408 torsion balance observational points and the site of the test area can be seen on Figure 3.

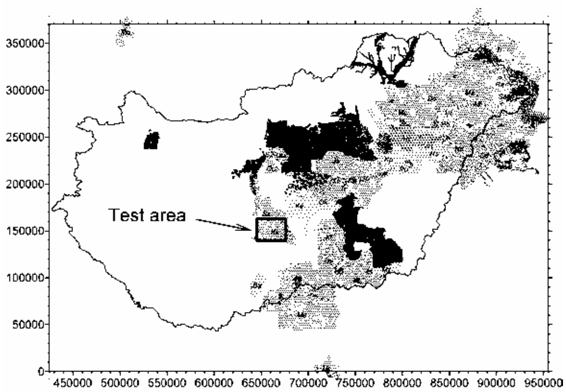

Fig. 3 Location of torsion balance measurements being stored in computer database, and the site of the test area

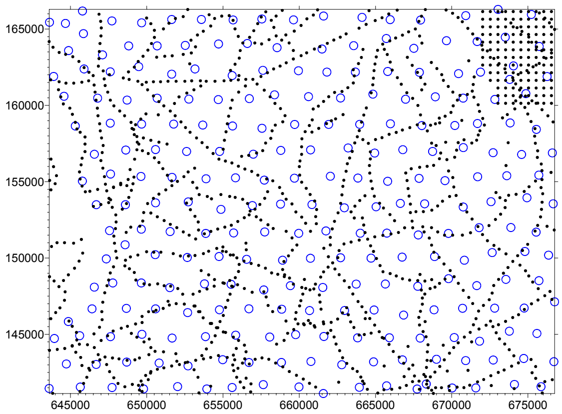

Fig. 4 Gravity measurements (marked by dots) and torsion balance points (marked by circles) on the test area

The nearly flat test area can be found in the middle of the country, the height difference between the lowest and highest points is less than 20 m. There were 248 torsion balance stations and 1197 gravity measurements on this area. 30 points from these

248 torsion balance stations were chosen as fixed points where gravity anomalies Δg are known from gravity measurements and the unknown gravity anomalies were interpolated on the remaining 218 points. Location of torsion balance stations (marked by circles) and the gravity measurements (marked by dots) can be seen on Figure 4.

The isoline map of gravity anomalies $\Delta g = g - \gamma$ (γ is the normal gravity) constructed from 1197 g measurements can be seen on Figure 5. Small dots indicate the locations of measured gravity values. Measurements were made by Worden gravimeters, by accuracy of ±20-30 μGal. At the same time the isoline map of gravity anomalies constructed from the interpolated values from 248 torsion balance measurements can be seen on Figure 6. Small circles indicate the locations of torsion balance points.

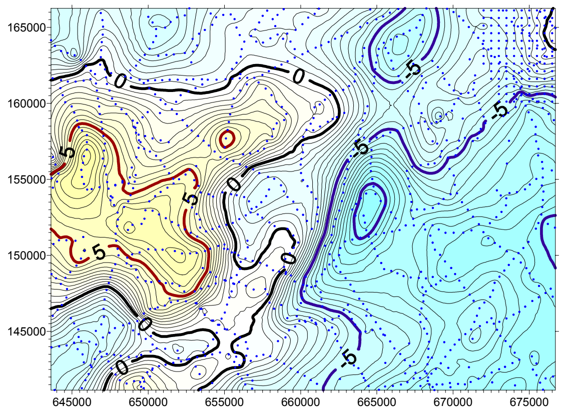

Fig. 5 Gravity anomalies from g measurements on the test area

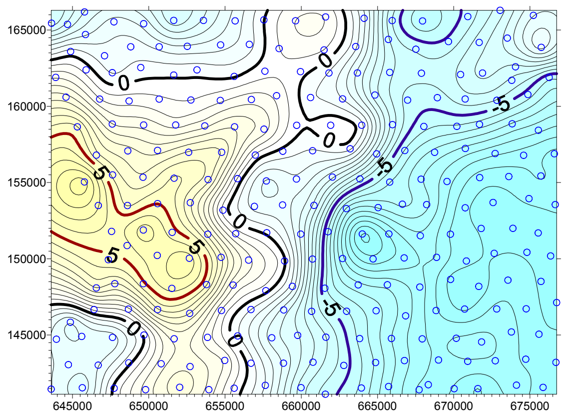

Fig. 6 Interpolated gravity anomalies from W_{zx} and W_{zy} gradients measured by torsion balance on the test area

More or less a good agreement can be seen between these two isoline maps. In order to control the applicability and accuracy of interpolation, we compared the given and the interpolated gravity anomalies. Δg values were determined for each torsion balance points from gravity measurements by linear interpolation on the one hand and gravity anomalies for the same points from gravity gradients measured by torsion balance on the other hand. Isoline and surface maps of differences between the two types of Δg values can be seen on Figures 7 and 8. The differences are about ±1–2 mGal the maximum difference is 4 mGal.

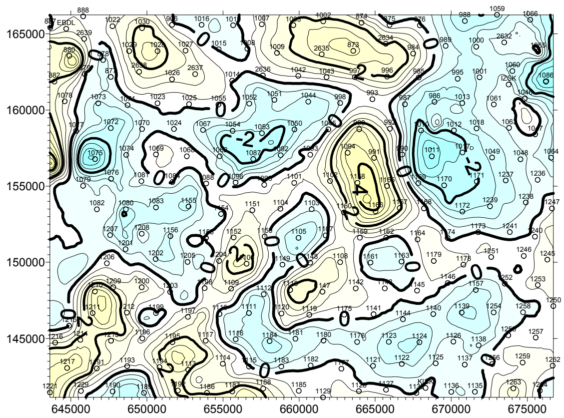

Fig. 7 Isoline map of differences between the measured and the interpolated gravity anomalies on the test area

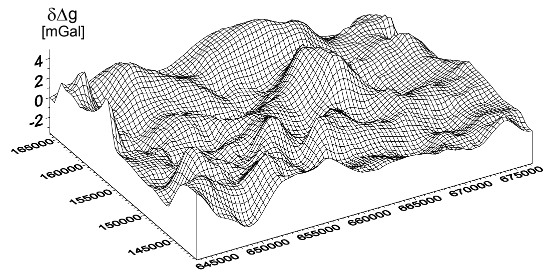

Fig. 8 Surface map of differences between the measured and the interpolated gravity anomalies on the test area

Finally the standard error characteristic to interpolation, determined by

$$m_{\Delta g} = \pm\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\Delta g_i^{mes.} - \Delta g_i^{\mathrm{int.}})^2}\ ,$$

was computed (where $\Delta g_i^{mes.}$ is the gravity anomaly from gravity measurements, $\Delta g_i^{\mathrm{int.}}$ is the interpolated value from torsion balance measurements and $n = 248$ is the number of torsion balance stations).

Standard error $m_{\Delta g} = \pm 1.281\,\mathrm{mGal}$ indicates that horizontal gradients of gravity give a possibility to determine gravity anomalies from torsion balance measurements by *mGal* accuracy on flat areas.

In case of a not quite flat area (like our test area) accuracy of interpolation would probably be increased by taking into consideration the effect of vertical gradients by integral (4), but unfortunately we haven't got the real vertical gradient values of torsion balance points on our test area yet. It would be important to investigate the effect of vertical gradient for the interpolation in the future.

Summary

A method was developed, based on integration of horizontal gradients of gravity W_{zx} and W_{zy}, to predict gravity anomalies at all points of the torsion balance network. Test computations were performed in a characteristic flat area in Hungary where both torsion balance and gravimetric measurements are available. Comparison of the measured and the interpolated gravity anomalies indicates that horizontal gradients of gravity give a possibility to determine gravity anomalies from torsion balance measurements by *mGal* accuracy on flat areas. Accuracy of interpolation would probably be increased by taking into consideration the effect of vertical gradients.

Acknowledgements

We should thank for the funding of the above investigations to the National Scientific Research Fund (OTKA T-037929 and T-37880), and for the assistance provided by the Physical Geodesy and Geodynamic Research Group of the Hungarian Academy of Sciences.

References

Detrekői Á. (1991) Adjustment calculations. Tankönyvkiadó, Budapest. (in Hungarian)

Heiskanen W, Moritz H. (1967) Physical Geodesy. W.H. Freeman and Company, San Francisco and London.

Torge W. (1989) Gravimetry. Walter de Gruyter, Berlin – New York.

Völgyesi L. (1993) *Interpolation of Deflection of the Vertical Based on Gravity Gradients*. Periodica Polytechnica Civ.Eng., Vo1. 37. Nr. 2, pp. 137-166.

Völgyesi L. (1995) *Test Interpolation of Deflection of the Vertical in Hungary Based on Gravity Gradients*. Periodica Polytechnica Civ.Eng., Vo1. 39, Nr. 1, pp. 37-75.

Völgyesi L. (2001) *Geodetic applications of torsion balance measurements in Hungary.* Reports on Geodesy, Warsaw University of Technology, Vol. 57, Nr. 2, pp. 203-212.

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Völgyesi L, Tóth Gy, Csapó G (2004): Determination of gravity anomalies from torsion balance measurements. IAG International Symposium, Gravity, Geoid and Space Missions. Porto, Portugal August 30 - September 3, 2004.

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